

# Economic principle on fuzzy profit by weighted average value

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## Abstract

The well-known economic principle on profit states that the profit is maximum when the marginal revenue equals the marginal cost. We hereby present the case where the demand and the cost are polynomials in the demand quantity variable. The coefficients are trapezoidal fuzzy numbers, hence the demand and the cost are fuzzy numbers too. Since our goal is *maximizing* the profit, we have to choose a suitable defuzzification method of fuzzy numbers. The method we use is the Weighted Average Value, which is more general than others presented by several authors. The results we obtain are a generalization of the crisp case.

## 1. Introduction

The classical mathematics is used in economy to detect rules of economy systems and to present these rules in a form of equations or mathematics models, which can be used by decision-makers for supporting decision process. The most important problem with the classical mathematics is that it requires reliable precise numeric data to create well working models. Unfortunately a lot of information, which can be acquired from official sources is not reliable at all. Moreover, a lot of information is also not measurable in a numeric way. This kind of information gives qualitative knowledge about the analysed system and therefore allows for deeper insight into the system.

In spite of being regularly taken into account by decision-makers who are able to estimate it using linguistic terms, the qualitative information is very often omitted in classical models. It is a result of classical mathematics application, which cannot fully utilized this kind of information. It is not a single case when qualitative information occurs to be more significant for real economic process than quantitative one but it is missing because of classical models limitations. Hence, the further progress in building models of economic systems and applying them in the decision making process depends on proper combination of quantitative economic knowledge with qualitative knowledge and introduction of the linguistic terms. This connection is possible via the idea of fuzzy logic. At the beginning fuzzy logic was used mainly in technical application. Recently the fuzzy logic has been successfully applied not only in technical applications but also in economical and social ones.

Many mathematical models for the real world use coefficients that are supposed to have fixed characteristics. Unfortunately, these parameters are often

unknown exactly because they are variable, unreliable or imprecise in some way. In several cases the coefficients involved in the description may only be approximately estimated by means of observations that are done at discrete steps or are obtained by a regression having the discrete observations as fixed points. This imperfect knowledge would normally be introduced by means the substitution of crisp values by fuzzy ones.

In all classical texts of Economics (i.e. [11]) the economic principle on profit, in the crisp sense, states that profit is maximum when marginal revenue equals marginal cost. A paper of Yao-Chang [13] faces this problem in a fuzzy contest. They examine several cases of cost and demand functions. The wider case is the one in which the functions involved are quadratic. The fuzzy quantities they propose are triangular fuzzy numbers (shortly, f.n.). They use the centroid as defuzzification method and show that the crisp principle can be naturally extended in a fuzzy context. The aim of our paper is to prove that the economic principle on profit is still valid even if the coefficients are modelled by to Trapezoidal fuzzy numbers (TR. f.n). They could be interpreted as a possibility distribution on the values, which any given coefficient may assume. We choose a trapezoidal fuzzy number as general framework for the description of vagueness and imprecision as they are easy to handle (it is defined by no ore than four paramethers) and have a natural interpretation. Recent papers have shown how, given a general fuzzy number, it is possible to find a trapezoidal approximation that fulfils important properties. [ 9] As defuzzification method we propose a more general one, which contains the centroid as particular case.

The method we use consists in the definition of a mapping from fuzzy numbers into the real line to obtain a suitable real value we call the defuzzification of the fuzzy number. The mapping, called Weighted Average Value (WAV), is introduced by Campos- Gonzales ([3, 4]), and presented in a detailed way in [6]. WAV depends on two parameters, a real number  $I$  and a probability measure  $S$ .  $I$  is connected with the pessimistic or optimistic point of view of the decision maker,  $S$  is connected with the preference of the decision maker to choose, according to his preference, different subsets of the support. The choice may depend on subjective elements, on the nature of the problem and on the sensitivity of the decision-maker. Even in this general scenario, the economic principle on profit is anyway valid and the solution of the fuzzy case has the same shape than the one obtained in the crisp case and collapses into the crisp solution when the fuzzy coefficients become crisp.

## 2. Economic principle on profit with crisp and fuzzy revenue and cost functions. An unitarian vision.

The classical economic principle on profit says that if the point  $x_0$  is the solution of the profit maximization, then it realizes the equality between the marginal revenue and the marginal cost. This value helps the monopolist to determine the unit price of the material he produces.

The functions here involved are polynomials in the  $x$  variable with real coefficients:

$$\Psi_i(a_{ij}, x) : R \times R \rightarrow R : \Psi_i(x) = \sum_{j=0}^n a_{ij} x^j, i=1 \dots 4.$$

In particular let, the demand function  $\Psi_1(x) = p(x) : [0, x_c] \rightarrow R$ , where the domain is such that  $p|_{[0, x_c]}(x) \geq 0$ , the total revenue  $\Psi_2(x) = R(x) = xp(x) : [0, x_c] \rightarrow R$ , the

total cost  $\Psi_3(x) = \Pi(x) : [0, +\infty) \rightarrow R$ , and the profit  $\Psi_4(x) = P(x) = R(x) - \Pi(x) : [0, x_c] \rightarrow R$ , be all polynomials in the  $x$  variable with real coefficients with some compatibility conditions on the coefficients.

To face the economic principle on profit problem, we have to find the points of maximum of the profit function, that is we have to find the critical points of  $P(x)$

$$\frac{d\Psi_4(x)}{dx} = \frac{dP(x)}{dx} = \frac{dR(x)}{dx} - \frac{d\Pi(x)}{dx} = 0,$$

that is

$$a_{10} + \sum_{j=1}^n (a_{1j}x^j - ja_{3j}x^{j-1}) = 0 \quad (2.1)$$

we have to solve an equation of grade  $n-1$  and to choose the points of maximum, we

have to verify for which  $\bar{x}_i$  the inequality  $\left. \frac{d^2P(x)}{dx^2} \right|_{x=\bar{x}_i} \geq 0$  is true. We will have cases

in which the problem is solvable in an explicit way and others in which this is not. It depends on  $n$ .

We propose a translation to the fuzzy case modifying all the coefficients of the polynomials involved from crisp values to Trapezoidal fuzzy numbers (TR.f.n). This fact produces that all the functions are real polynomials with fuzzy coefficients and are TR.f.n. We choose a trapezoidal fuzzy number as general framework for the description of vagueness and imprecision as they are easy to handle (it is defined by no more than four parameters) and have a natural interpretation. Recent papers have shown how given a general fuzzy number it is possible to find a trapezoidal approximation that fulfils important properties. [ 9]

The last step is the maximizing step. We may follow two procedures:

- We may defuzzify the fuzzy profit and then find the points of maximum of the real function we obtain.
- We may defuzzify the derivative of fuzzy profit and then find the solutions of this real equation.

If the defuzzification method is linear in the four parameters which describe the TR.f.ns, the two steps are equivalent, if it is not we have to follow the first procedure.

In this paper we follow the first one and propose to use a general defuzzification method that contains, as particular cases, a wide number of known methods present in literature: the weighted average value (WAV) [see 1-8,12 ].

### 3. Weigthed Average value of fuzzy numbers

We remember that a fuzzy set  $\tilde{u}$  is defined by a generalized characteristic function  $m_{\tilde{u}}(\cdot)$ , called membership function, defined on a universe  $X$ , which assumes values in  $[0,1]$ . The universe, in a concrete case, should be chosen according to the specific situation. In the following,  $X$  denotes a non empty subset of  $R$ .

**Definition 3.1.** The fuzzy set  $\tilde{u}$  is a fuzzy number iff:

- 1)  $\forall a \in [0,1], u_a = \{x \in R : m_{\tilde{u}}(x) \geq a\} = [a_1^a, a_2^a]$ , called  $a$ -cuts of  $\tilde{u}$ , is a convex set.
- 2)  $m_{\tilde{u}}(\cdot)$  is an upper-semicontinuous function.

3)  $Supp(\tilde{u}) = \{x \in R : \mu_{\tilde{u}}(x) > 0\}$  is a bounded set in  $R$ .

4)  $\exists x \in Supp(\tilde{u})$  s.t.  $\mathbf{m}_{\tilde{u}}(x) = 1$

We denote the set of fuzzy numbers by  $F$ .

**Definition 3.2.** We call *Weighted Average Value* (WAV) of the fuzzy number  $\tilde{u}$ , by the measure  $S$  on  $[0,1]$ , the *F-evaluation function*

$$\overline{M}_I(S, \tilde{u}) = \int_0^1 (I u_2^a + (1-I) u_1^a) dS \quad (3.1)$$

where  $u^a = [u_1^a, u_2^a]$  is an  $\mathbf{a}$ -cut of  $\tilde{u}$  and the parameter  $I \in [0,1]$  may be interpreted as an optimistic or pessimistic point of view. Notice that it results

$$\overline{M}_I(S, \tilde{u}) = I M^*(S, \tilde{u}) + (1-I) M_*(S, \tilde{u}) \quad (3.2)$$

having

$$M_*(S, \tilde{u}) = \int_0^1 u_1^a dS \quad \text{and} \quad M^*(S, \tilde{u}) = \int_0^1 u_2^a dS \quad (3.3)$$

The interval  $\overline{M}(S, \tilde{u}) = [M_*(S, \tilde{u}), M^*(S, \tilde{u})]$  is called the *S-mean value* of  $\tilde{u}$ .

If the fuzzy number  $\tilde{u}$  is the crisp real number  $u$ ,  $\overline{M}_I(S, u) = u$ .

If  $S$  is the Stieltjes measure generated by the function  $s(\mathbf{a}) = \mathbf{a}^r$ ,  $\forall r > 0$ :  $S([a, b]) = b^r - a^r$ ,  $\forall a, b \in [0,1]$ , it is possible to write the *F-evaluation function* in (3.2) in a simpler way:

$$\overline{M}_I(S, \tilde{u}) = I M^*(r, \tilde{u}) + (1-I) M_*(r, \tilde{u})$$

where

$$M_*(r, \tilde{u}) = r \int_0^1 \mathbf{a}^{r-1} u_1^a d\mathbf{a} \quad \text{and} \quad M^*(r, \tilde{u}) = r \int_0^1 \mathbf{a}^{r-1} u_2^a d\mathbf{a} \quad (3.4)$$

We call  $\overline{M}_I(r, \tilde{u})$  the *Weighted Average Value of  $\tilde{u}$  of order  $r$* .

It is easy to see that for particular choices of  $I$  and  $S$ , the (3.4) coincides with other comparison indexes (see 1-8,12). For more details see [6].

The geometric meaning of  $r$  is connecting with this observation. Denote by  $\tilde{u}^r$  the fuzzy number with membership function  $\mathbf{m}_{\tilde{u}^r} = x^r \circ \mathbf{m}_{\tilde{u}}$ .

We have

$$[M_*(r, \tilde{u}), M^*(r, \tilde{u})] = [M_*(1, \tilde{u}^r), M^*(1, \tilde{u}^r)].$$

If  $\tilde{u}$  is a TR f.n., the previous interval is a subset of the support of  $\tilde{u}$  which is the projection on the real axe of the segment we obtain cutting  $\tilde{u}$  at the level

$$\bar{a} = \int_0^1 \mathbf{a}^r d\mathbf{a} = \frac{1}{r+1}.$$

A consequence of this choice is that the defuzzified value lies in a subset of the TR.f.n support, which becomes as much narrow as the value of  $r$  increases.

**Definition 3.3.** A TR.f.n.  $\tilde{u} = (u_1 - \Delta_1, u_1, u_2, u_2 + \Delta_2)$ , where  $\Delta_1 > 0$ ,  $\Delta_2 > 0$ , is represented by the membership function

$$m_{\tilde{u}}(x) = \begin{cases} 0 & x \leq u_1 - \Delta_1 \\ 1 + \frac{x - u_1}{\Delta_1} & u_1 - \Delta_1 < x \leq u_1 \\ 1 & u_1 < x \leq u_2 \\ 1 + \frac{u_2 - x}{\Delta_2} & u_2 < x \leq u_2 + \Delta_2 \\ 0 & x > u_2 + \Delta_2 \end{cases}$$

If  $u_1 = u_2$ , then  $\tilde{u}$  is a triangular fuzzy number.

In this case we have

$$\begin{aligned} \bar{M}_1(r, \tilde{u}) &= I M^*(r, \tilde{u}) + (1 - I) M_*(r, \tilde{u}) = \\ I u_2 + (1 - I) u_1 + \frac{(I \Delta_2 - (1 - I) \Delta_1)}{(r + 1)} \end{aligned} \quad (3.5)$$

If  $\Delta_1 = \Delta_2$  and  $I = \frac{1}{2}$ ,  $\bar{M}(r, \tilde{u}) = \frac{u_1 + u_2}{2}$ .

That is, the *Average Value of order r* of a symmetric trapezoidal fuzzy number is the central value of the flat part, if and only if the weight is  $I = \frac{1}{2}$ .

The definition of *Weighted Average Value of  $\tilde{u}$  of order r* will be used as a defuzzification method.

#### 4. Economic principle on profit with crisp revenue and cost function

As we desire to solve in an explicit way the (2.1) we consider the more general crisp case in which this is possible: quadratic demand and cost functions,

$$p(x) = a - bx - cx^2, \quad \text{and} \quad \Pi(x) = d + ex + gx^2, \quad (4.1)$$

where all coefficients are real numbers such that  $a > d > 0$ ,  $a > e > 0$ ,  $b > 0$ ,  $c > 0$ ,  $g > 0$  and

$$0 \leq x \leq \hat{x}, \quad \hat{x} = \frac{-b + \sqrt{b^2 + 4ac}}{2c}. \quad (4.2)$$

The hypothesis on the coefficients causes that the two functions  $p(x)$  and  $\Pi(x)$  are respectively positive and decreasing, positive and increasing for every  $x \in [0, \hat{x}]$ .

If  $R(x) = xp(x)$  is the total revenue function, and  $x$  are the units of the sold commodities, the profit function is

$$P(x) = R(x) - \Pi(x) = -3cx^2 - 2(b + g)x - (e - a) \quad (4.3)$$

$P(x)$  is concave on the domain  $[0, \hat{x}]$ , as

$$P'(x) = +3(-c)x^2 + 2(-b - g)x + (a - e); \quad P''(x) = 6(-c)x + 2(-b - g) < 0.$$

Consequently, the maximum point  $x_0$  of  $P(x)$  is simply the positive solution of the equation  $P'(x) = 0$  (which we call *characteristic equation of the crisp problem*), that is

$$x_0 = \frac{-(b+g) + \sqrt{(b+g)^2 + 3c(a-e)}}{3c}. \quad (4.4)$$

This solution always exists by the assumption we made:  $(a-e) > 0$ . It can easily be shown that if  $x_0 < \hat{x}$  then the solution is admissible and is the solution of the economic principle on profit.

## 5. Economic principle on profit with revenue and cost fuzzy functions

If we wish to model a real situation, we have to note that coefficients in demand and cost functions (2.1) are not exactly given: they are obtained by estimations that cannot be based on probabilistic assumptions, but only by the opinions of economic experts. For instance, the  $d$  value (i.e. the fixed cost) may be estimated in an imprecise way and the experts could assert: “We think that the fixed costs will be about in the interval  $[d_1, d_2]$ ”. This “vague” information may be written in a fuzzy sense expressing all coefficients by means of trapezoidal fuzzy numbers which better translate the linguistic description of the coefficients.

Let  $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{g}$  be trapezoidal fuzzy numbers, that is

$$\begin{aligned} \tilde{a} &= (a_1 - \Delta_{11}, a_1, a_2, a_2 + \Delta_{12}), & \tilde{b} &= (b_1 - \Delta_{21}, b_1, b_2, b_2 + \Delta_{22}), \\ \tilde{c} &= (c_1 - \Delta_{31}, c_1, c_2, c_2 + \Delta_{32}), & \tilde{d} &= (d_1 - \Delta_{41}, d_1, d_2, d_2 + \Delta_{42}), \\ \tilde{e} &= (e_1 - \Delta_{51}, e_1, e_2, e_2 + \Delta_{52}), & \tilde{g} &= (g_1 - \Delta_{61}, g_1, g_2, g_2 + \Delta_{62}), \end{aligned} \quad (5.1)$$

such that

$$\begin{aligned} a_1 < a < a_2, \quad b_1 < b < b_2, \quad c_1 < c < c_2, \quad d_1 < d < d_2, \quad e_1 < e < e_2, \quad g_1 < g < g_2 \\ 0 < \Delta_{i1}, \quad 0 < \Delta_{i2} \quad \forall i = 1, \dots, 6 \\ \Delta_{11} < a_1, \quad \Delta_{21} < b_1, \quad \Delta_{31} < c_1, \quad \Delta_{41} < d_1, \quad \Delta_{51} < e_1, \quad \Delta_{61} < g_1 \end{aligned} \quad (5.2)$$

(shortly, these inequalities mean that the trapezoidal f.n. are positive)

The demand function

$$\tilde{p}(x) = \tilde{a} - \tilde{b}x - \tilde{c}x^2 \quad (5.3)$$

and the cost function

$$\tilde{\Pi}(x) = \tilde{d} + \tilde{e}x + \tilde{g}x^2 \quad (5.4)$$

can be defined for every demand quantity  $x > 0$ .

If  $T$  is the set of trapezoidal fuzzy numbers, by the usual arithmetic operations in  $T$  we have that the fuzzy demand  $\tilde{p}(x) = \tilde{a} - \tilde{b}x - \tilde{c}x^2$  is again a trapezoidal fuzzy number

$$\tilde{p}(x) = (S_1, S_2, S_3, S_4),$$

$$\begin{aligned} S_1 &= a_1 - \Delta_{11} - (b_2 + \Delta_{22})x - (c_2 + \Delta_{32})x^2 \\ S_2 &= a_1 - b_2x - c_2x^2, \quad S_3 = a_2 - b_1x - c_1x^2, \\ S_4 &= a_2 + \Delta_{12} - (b_1 - \Delta_{21})x - (c_1 - \Delta_{31})x^2 \end{aligned} \quad (5.5)$$

Notice that since  $\tilde{p}(x)$  and  $\tilde{\Pi}(x)$  have the same economic meaning given in the crisp sense, it is necessary that the fuzzy numbers  $\tilde{p}(x)$  and  $\tilde{\Pi}(x)$  are positive.

The positivity of  $\tilde{\Pi}(x)$  for every  $x > 0$  is immediate as  $\tilde{d}, \tilde{e}, \tilde{g}$  are positive TR.f. n. by (5.1). Consider now the fuzzy demand  $\tilde{p}(x)$ : it will be positive if and only if

$$S_1 = a_1 - \Delta_{11} - (b_2 + \Delta_{22})x - (c_2 + \Delta_{32})x^2 > 0.$$

The last inequality will be true if

$$0 < x < \bar{x} = \frac{-(b_2 + \Delta_{22}) + \sqrt{(b_2 + \Delta_{22})^2 + 4(a_1 - \Delta_{11})(c_2 + \Delta_{32})}}{2(c_2 + \Delta_{32})}. \quad (5.6)$$

Since we had a similar bound also in the crisp case (see (4,2)), it is interesting to investigate the connection between  $\bar{x}$  and the crisp bound  $\hat{x}$ , which can simply be obtained by  $\bar{x}$  considering  $\Delta_{11} = \Delta_{22} = \Delta_{32} = 0$  and

$$a_1 = a_2 = a, b_1 = b_2 = b, c_1 = c_2 = c.$$

For this purpose, we can study the function  $f(\Delta_{11}, \Delta_{22}, \Delta_{32}) : R_+^3 \rightarrow R$

$$f(\Delta_{11}, \Delta_{22}, \Delta_{32}) = \frac{-(b_2 + \Delta_{22}) + \sqrt{(b_2 + \Delta_{22})^2 + 4(a_1 - \Delta_{11})(c_2 + \Delta_{32})}}{2(c_2 + \Delta_{32})}.$$

It is easy to check that  $\frac{\partial f}{\partial \Delta_{11}} < 0$ ,  $\frac{\partial f}{\partial \Delta_{22}} < 0$ ,  $\frac{\partial f}{\partial \Delta_{32}} < 0$ .

$$\text{Hence} \quad \bar{x} = f(\Delta_{11}, \Delta_{22}, \Delta_{32}) \leq f(0, 0, 0) = \hat{x}$$

and  $\tilde{p}(x)$  is positive for every  $x \in [0, \bar{x}] \subseteq [0, \hat{x}]$ .

We can therefore calculate the revenue (fuzzy) function, for every  $x \in [0, \bar{x}]$ ,

$$\tilde{R}(x) = x \tilde{p}(x) = x(\tilde{a} - \tilde{b}x - \tilde{c}x^2) = (R_1, R_2, R_3, R_4),$$

$$R_1 = (a_1 - \Delta_{11})x - (b_2 + \Delta_{22})x^2 - (c_2 + \Delta_{32})x^3$$

$$R_2 = a_1x - b_2x^2 - c_2x^3, \quad R_3 = a_2x - b_1x^2 - c_1x^3, \quad (5.7)$$

$$R_4 = (a_2 + \Delta_{12})x - (b_1 - \Delta_{21})x^2 - (c_1 - \Delta_{31})x^3$$

and the fuzzy cost function  $\tilde{\Pi}(x) = \tilde{d} + \tilde{e}x + \tilde{g}x^2 = (\Pi_1, \Pi_2, \Pi_3, \Pi_4)$  where

$$\Pi_1 = d_1 - \Delta_{41} + (e_1 - \Delta_{51})x + (g_1 - \Delta_{61})x^2,$$

$$\Pi_2 = d_1 + e_1x + g_1x^2, \quad \Pi_3 = d_2 + e_2x + g_2x^2, \quad (5.8)$$

$$\Pi_4 = d_2 + \Delta_{42} + (e_2 + \Delta_{52})x + (g_2 + \Delta_{62})x^2.$$

Then, the fuzzy profit is the set

$$\tilde{P}(x) = \tilde{R}(x) - \tilde{\Pi}(x) = -\tilde{d} + \tilde{a}x - \tilde{e}x - \tilde{b}x^2 - \tilde{g}x^2 - \tilde{c}x^3 = (P_1, P_2, P_3, P_4)$$

where, by (4.7) and (4.8),

$$P_1 = -(d_2 + \Delta_{42}) + (a_1 - \Delta_{11} - e_2 - \Delta_{52})x - (b_2 + \Delta_{22} + g_2 + \Delta_{62})x^2 - (c_2 + \Delta_{32})x^3$$

$$P_2 = -d_2 + (a_1 - e_2)x - (b_2 + g_2)x^2 - c_2x^3,$$

$$P_3 = -d_1 + (a_2 - e_1)x - (b_1 + g_1)x^2 - c_1x^3 \quad (5.9)$$

$$P_4 = -(d_1 - \Delta_{41}) + (a_2 + \Delta_{12} - e_1 + \Delta_{51})x - (b_1 - \Delta_{21} + g_1 - \Delta_{61})x^2 - (c_1 - \Delta_{31})x^3.$$

Let us consider on  $T$  the defuzzification method defined by the *Weighted Average Value of order  $r$*  (WAV) as follows from (3.5) :

$$\bar{M}_I(r, \tilde{P}) = IP_3 + (1-I)P_2 + \frac{\lambda(P_4 - P_3) - (1-\lambda)(P_2 - P_1)}{(r+1)}.$$

As we are interested in maximizing the WAV of the fuzzy profit, let us calculate

$$\begin{aligned}\frac{d}{dx} \bar{M}_I(r, \tilde{P}(x)) &= \frac{d}{dx} \left( I P_3 + (1-I) P_2 + \frac{I(P_4 - P_3) - (1-I)(P_2 - P_1)}{(r+1)} \right) \\ &= I P_3' + (1-I) P_2' + \frac{I(P_4' - P_3') - (1-I)(P_2' - P_1')}{(r+1)}\end{aligned}$$

where  $P_i'(x) = \frac{dP_i}{dx}$  for ever  $i=1,2,3,4$ , are:

$$\begin{aligned}P_1' &= (a_1 - \Delta_{11} - e_2 - \Delta_{52}) - 2(b_2 + \Delta_{22} + g_2 + \Delta_{62})x - 3(c_2 + \Delta_{32})x^2 \\ P_2' &= (a_1 - e_2) - 2(b_2 + g_2)x - 3c_2x^2 \\ P_3' &= (a_2 - e_1) - 2(b_1 + g_1)x - 3c_1x^2 \\ P_4' &= (a_2 + \Delta_{12} - e_1 + \Delta_{51}) - 2(b_1 - \Delta_{21} + g_1 - \Delta_{61})x - 3(c_1 - \Delta_{31})x^2.\end{aligned}\quad (5.10)$$

Consequently,

$$\begin{aligned}\frac{d}{dx} \bar{M}_I(r, \tilde{P}(x)) &= I[(a_2 - e_1) - 2(b_1 + g_1)x - 3c_1x^2] + \\ & (1-I)[(a_1 - e_2) - 2(b_2 + g_2)x - 3c_2x^2] \\ & + \frac{1}{r+1} \left\{ I[\Delta_{12} + \Delta_{51} + 2x(\Delta_{21} + \Delta_{61}) + 3x^2\Delta_{31}] - (1-I)[\Delta_{11} + \Delta_{52} + 2x(\Delta_{22} + \Delta_{62}) + 3x^2\Delta_{32}] \right\} \\ & = -\left( 3\mathbf{a}x^2 + 2\mathbf{b}x + \mathbf{g} \right)\end{aligned}\quad (5.11)$$

having denoted

$$\begin{aligned}\mathbf{a} &= I c_1 + (1-I) c_2 + \frac{(1-I)\Delta_{3,2} - I\Delta_{3,1}}{r+1} = M_{1-I}(r, \tilde{c}) \\ \mathbf{b} &= I(b_1 + c_1) + (1-I)(b_2 + c_2) + \frac{I(\Delta_{2,1} + \Delta_{6,1}) - (1-I)(\Delta_{2,2} + \Delta_{6,2})}{r+1} = \\ & M_{1-I}(r, \tilde{b} + \tilde{g}) \\ \mathbf{g} &= I(e_1 - a_2) + (1-I)(e_2 - a_1) + \frac{(1-I)(\Delta_{1,1} + \Delta_{5,2}) - I(\Delta_{1,2} + \Delta_{5,1})}{r+1} = M_{1-I}(r, \tilde{e} - \tilde{a})\end{aligned}$$

that is

$$\frac{d}{dx} \bar{M}_I(r, \tilde{P}(x)) = -3M_{1-I}(r, \tilde{c})x^2 - 2M_{1-I}(r, \tilde{b} + \tilde{g})x - M_{1-I}(r, \tilde{e} - \tilde{a})$$

We call *characteristic equation of the fuzzy problem* the equation

$$M_{1-I}(r, \tilde{c})x^2 + 2M_{1-I}(r, \tilde{b} + \tilde{g})x + M_{1-I}(r, \tilde{e} - \tilde{a}) = 0 \quad (5.12)$$

It is easy to check that the last equation has exactly the same shape than the characteristic equation (4.3) of the crisp case, provided that we change each crisp coefficient with the WAV of the corresponding fuzzy coefficient. Consequently, we can proceed through the same path and obtain

$$\begin{aligned}\tilde{x} &= \frac{-M_{1-I}(r, \tilde{b} + \tilde{g}) \pm \sqrt{M_{1-I}^2(r, \tilde{b} + \tilde{g}) - 3M_{1-I}(r, \tilde{c})M_{1-I}(\tilde{e} - \tilde{a})}}{3M_{1-I}(r, \tilde{c})} = \\ & \frac{-\mathbf{b} \pm \sqrt{\mathbf{b}^2 - 3\mathbf{a}\mathbf{g}}}{3\mathbf{a}}\end{aligned}\quad (5.13)$$



that has the same shape than the crisp solution  $x_0$  (see (4.4)). If the fuzzy coefficients are crisp numbers, then  $\tilde{x} = x_0$ .

As  $\frac{d^2}{dx^2} \bar{M}_I(r, \tilde{P}(x)) = -2(3ax + b) < 0$ ,  $\bar{M}_I(r, \tilde{P}(x))$  is a concave function and as we want positive solutions, by the inequalities  $a > 0, b > 0$ , it is easy to note that the unique case in which this happens is :  $M_{1-I}(r, \tilde{e} - \tilde{a}) = g < 0$ .

In fact if  $g < 0$ , using the previous notations, the discriminant  $\Omega = b^2 - 3ag$  is positive and we obtain two roots, one positive and one negative.

The one we need is the positive one that is

$$\tilde{x} = \frac{-b + \sqrt{b^2 - 3ag}}{3a} \quad (5.14)$$

if  $\tilde{x} < \bar{x}$ , the solution is admissible.

In the other two cases,  $g = 0, g > 0$ , we have respectively one negative solution and two negative solutions that are not admissible as we have  $x > 0$ .

What does it mean  $g < 0$ ? For the properties of WAV we have that

$$g = M_{1-I}(r, \tilde{e} - \tilde{a}) = -M_I(r, \tilde{a} - \tilde{e}) = M_{1-I}(r, \tilde{e}) - M_I(r, \tilde{a}) < 0$$

that is  $M_{1-I}(r, \tilde{e}) < M_I(r, \tilde{a})$ .

In the crisp case we have the hypothesis of  $e < a$ . Here we need of a stronger hypothesis: As we may choose  $I \in [0, 1]$ , we need that  $e_2 + \Delta_{5,2} < a_1 - \Delta_{1,1}$ , that is the supports of  $\tilde{e}$  and  $\tilde{a}$  are disjoint and  $\text{supp } \tilde{e} < \text{supp } \tilde{a}$ , where  $<$  is the usual order relation on intervals.

## Conclusions

The obtained results show that, using a general defuzzification method as WAV, it is possible to translate the crisp problem of economic principle on profit in a fuzzy sense, without losing anything. We find an admissible solution that collapses to the crisp solution when fuzzy coefficients become crisp coefficients.

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