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# Identification, estimation of multivariate transfer functions

di

Carlo Grillenzoni
Università di Modena

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Dipartimento di Economia Politica
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41100 Modena (Italy)
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# IDENTIFICATION, ESTIMATION of MULTIVARIATE TRANSFER FUNCTIONS

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Carlo Grillenzoni

*(University of Modena)*

Abstract - The problems of identification and estimation of a closed-loop system of simultaneous transfer function models are considered with the approach of the "stochastic approximation". The classical properties of stability, structural identification and realization of the system are preliminarily investigated, then a theoretical-practical method of obtaining simplified moving average representation and covariance factorization is defined . On the resulting expressions we have derived a disaggregate strategy of system identification which directly extends the univariate-unidirectional Box-Jenkins technics. In the estimation context the classical non-linear least squares estimators are considerably simplified by approximating, as in the recursive pseudolinear regression algorithm, the gradient with the input-output quantities of the system. Finally in an extended empirical example we have checked the validity of the approximate representations, of the approximate estimators and we have compared the statistical performance of the transfer function system with that of the vector ARMA model .

Keywords - Stochastic approximation, Simplified MA decomposition, Vector ARMA model, Iterative pseudolinear regression.

## 1. INTRODUCTION

This paper deals with the analysis of the structure of a system of simultaneous transfer functions (TFS), with special reference to the practical implications on the methods of identification and estimation . The approach followed is that of "*stochastic approximation*" and in this section we introduce it .

A model building methodology for open-loop rational transfer functions as

$$(TF) \qquad y_t = v(B)\, x_{t-b} + \psi(B)\, a_t$$

$$v(B) = \frac{(\omega_0 - \omega_1 B - \ldots \omega_s B^s)}{(1-\delta_1 B - \ldots - \delta_r B^r)}\,, \qquad \psi(B) = \frac{(1-\theta_1 B - \ldots - \theta_q B^q)}{(1-\phi_1 B - \ldots - \phi_p B^p)}\,,$$

$y_t$ = output process
$x_t$ = input process
$a_t$ = white disturbance
$B$ = backward operator

has been provided by Box-Jenkins(1970). In particular, a non-parametric strategy of identification has been developed on the second order properties of the system , expressed by the covariance functions (CVF)

$$(CCVF) \qquad \gamma_{xy}(B) = v(B)\, B^b\, \gamma_{xx}(B) \qquad \text{(cross)}$$

$$(ACVF) \qquad \gamma_{xx}(B) = \psi_x(B)\, \psi_x(F)\, \sigma^2 \qquad \text{(auto)}\,, \qquad F = 1/B$$

The advantages of the Box-Jenkins strategy lie in the simplicity of the moment estimators of the $\gamma(B)$, in their ability to treat the system functions $v(B)$, $\psi(B)$ separately and in providing a concise view of their whole dynamic.

A multivariate closed-loop extension of the TF model is given by the system

$$(TFS) \qquad \mathbb{z}_t = \mathbb{V}(B)\, \mathbb{z}_t + \dot{\Psi}(B)\, \mathbb{a}_t \qquad \text{(rational)}$$

$$\mathbb{V}(B) = \{v_{ij}(B)\, B^{b_{ij}}\}\,, \qquad \dot{\Psi}(B) = \mathrm{Diag}[\,\psi_{ii}(B)]$$

Although this model, through a linearization by row, can be reduced to a constrained multivatiate (m) ARMA system

$$(ARMA_m) \qquad \Phi(B)\, \mathbb{z}_t = \Theta(B)\, \mathbb{a}_t \qquad \text{(linear)}$$

$$\Phi(B) = \{\phi_{ij}(B)\} \quad , \qquad \Theta(B) = \{\theta_{ij}(B)\}$$

with $\Theta(B)$ diagonal, and although any $ARMA_m$ can be cast in rational form by matrix inversion, the TFS seems preferable for the following reasons :

1) the rational structure is more powerful and parsimonious of the linear one ;
2) the univariate residuals enable a simplified estimation by row ;
3) it has a *complete* realizatin theory from the spectral factorization theorem;
4) each impulse-response function $v_{ij}(B)$ has a proper denominator ;
5) it respects the different *nature* of the auto and cross dynamic regressions ;
6) potentially, it may be identified in a *disaggregate* manner by means of the Box-Jenkins technics .

The last point is the central topic of the paper. The major theoretical obstacle to it lies in the fact that the parametric expression of the covariance matrix is very complex

$$\Gamma(B) = [\mathbb{I} - \mathbb{V}(B)]^{-1}\dot{\Psi}(B)\dot{\Sigma}\,\dot{\Psi}(F)\;[\mathbb{I} - \mathbb{V}(F)']^{-1}$$

whereas for a disaggregate identification we should have

$$(1.1) \qquad \Gamma(B) \cong [\,\mathbb{I} + \mathbb{V}(B)]\;\;\dot{\Psi}(B)\dot{\Sigma}\dot{\Psi}(F) \;\Rightarrow\; \begin{array}{l} \gamma_{ij}(B) \cong v_{ij}(B)\;\gamma_{ii}(B) \\ \gamma_{ii}(B) \cong \psi_{ii}(B)\psi_{ii}(F)\sigma^2 \end{array}$$

To obtain the factorization (1.1), a necessary step will be to demonstrate the validity of the representation

$$(1.2) \qquad \mathbb{z}_t \cong [\,\dot{\Psi}(B) + \mathbb{V}(B)\,]\;\mathbb{a}_t$$

which, on the other hand, considerably simplifies the stability conditions .

Although the identification of TFS models by means of (1.1) has provided good empirical results, and although the inversion (1.2) is very sensible since, as in the univariate analysis, the number of parameters involved remains unchanged. To show the validity of (1.1), (1.2), we cannot simply use approximation arguments in the context of the conventional algebraic analysis; but we must also resort to nonconventional mathematical operators such as rational projectors on linear Hilbert spaces .

The second part of the paper deals with estimation methods of both TFS and

$ARMA_m$ models. In the present situation, packages for the joint estimation of simultaneous transfer function equations do not exist, moreover the existing routines for the $ARMA_m$ estimation encounter serious problems when m>2. Now, by extending at multivariate and iterative level the recursive algorithm known as pseudolinear regression (see Ljung-Söderström(1983)), we have defined estimators easily implementable on the existing statistical software.

The method works by approximating the gradient with input -output quantities $\{z_t, a_t\}$ ; the resulting estimators are asymptotically efficient and also mean square convergent if the condition of system polynomials "passive" holds. In many situations however, calculation is hindered by the necessity to have constant *stepsize* ($\frac{1}{2}$) to ensure convergence.

The paper ends with an extended example on 5 real economic time series. In it we check empirically identification and estimation methods, the structural properties of TFS and $ARMA_m$, finally we compare the statistical performance of the two class of models. The superiority of the TFS will be demonstrated .

*(1.1) Simplified AR-representation*

In this section we analyse conditions of orthogonality between the rational polynomials v(B), ψ(B), and their consequences in simplifying the identification and the estimation of the transfer function model .

Consider a rational transfer function and pass to its AR-representation

$$(1.3)\qquad y_t = v(B)B^b x_t + \psi(B)a_t$$

$$(1.4)\qquad \pi(B)y_t - w(B)B^b x_t = a_t \quad , \quad \pi(B) = \psi(B)^{-1}$$

from the univariate ARMA analysis , one would expect the number of parameters involved and/or the order of the rational polynomials not to change. Only the nature of the parameters (rational or linear) might be allowed to change.

In the strictly arithmetical sense however, the function $w(B)=v(B)B^b\pi(B)$ contains $(p+q)$ new parameters, and this is in contradiction with the sequential filtering mechanism implicit in (1.3)

$$\begin{cases} y_t - v(B)x_{t-b} = n_t \\ n_t = \begin{cases} \pi_1(B)n_t + a_t \\ \pi_1^*(B)y_t + a_t \end{cases} \end{cases} \qquad \pi_1(B)=[\pi(B) - 1]$$

where, in the first step CCV(y,x) is filtered independently on ACV(y), and in the second step since ACV(n) depends on ACV(y) the representation of $n_t$ may occur directly through the $\{y_{t-k}\}$ basis, by means of a suitable polynomial $\pi^*(B)$.

Now, the only possible way to have (1.3),(1.4) parametrically equivalent is to require some form of polynomial orthogonality such as

$$\pi(B)\ v(B)B^b = [1-\pi_1(B)]\ v_b(B) = v_b(B)$$

i.e. $$[\pi_1(B) = \Sigma_{k=1}^{\infty}\ \pi_k B^k] \perp [v_b(B) = \Sigma_{k=o}^{\infty} v_k B^{k+b}]$$

To this end, consider $\pi_1(B)$, $v_b(B)$ in the space $P(B)$ of the linear convergent polynomials of B . $P(B)$ is a linear vector space and the sequences $\{\pi_k\}$, $\{v_k\}$ represent the coordinates of $\pi_1(B)$, $v_b(B)$ on the $\{B^k\}$ axes. A measure of polynomial orthogonality is then provided by the inner product

$$((\pi_1, v_b)) = [\pi_b, \pi_{b+1} \ \ldots\ ]\ [v_o, v_1, v_2\ \ldots\ ]'$$

In practical terms, the above measure is close to zero assuming :

i) $\{v_k\}$,$\{\pi_k\}$ decaying rapidly (i.e. adequate stability),

ii) a *delay factor* (b) relatively high ,

iii) $\{v_k\}$,$\{\pi_k\}$ non-monotonic (e.g. complex roots).

To stress the role of (b), note that if $n_t \sim AR(p)$ with $p<b$, then $((\pi_1, v_b)) \equiv 0$.

This orthogonal property has important practical consequences. As we shall see later, it simplifies the computation of the gradient in the estimation phase; in the identification context, having the equivalent representations

$$y_t = \begin{cases} v_b(B)\ x_t + \pi_1(B)\ n_t + a_t \\ \pi_1(B)\ y_t + v_b(B)\ x_t + a_t \end{cases}$$

we conclude that ACV(y) ≡ ACV(n), hence the identification of $\psi(B)$ may occur on $\gamma_{yy}(B)$ directly .

*(1.2) Equivalence of PCCV - CCV*

Differently to the univariate ARMA analysis, Box-Jenkins(1970) have not introduced in the identification of $v_b(B)$ the partial cross correlation function (PCCRF) . This function may be simply defined as the sequence of marginal regres sion coefficients in dynamic regressions of increasing order

*(PCCRF)* $\{v_{kk}\}_o^\infty \;\varepsilon\; y_t = \Sigma_{j=o}^{k} v_{jk}\, x_{t-j} + n_t$

and can be computed through the deterministic system

$$\gamma_{xy}(i) = \Sigma_{j=o}^{k} v_{jk}\, \gamma_{xx}(j-i) \qquad i=0,1,2\ \ldots.$$

i.e.
$$\boldsymbol{\gamma}_{xy} = \boldsymbol{\Gamma}_{xx}\, \mathbf{v}_k$$

The unnecessity of $\{v_{kk}\}$ suggests that CCVF , PCCVF should have the same information and/or the same pattern and, in effect, for $\{x_t\}$ white noise ($\boldsymbol{\Gamma}_{xx}$ diagonal), we have $\boldsymbol{\gamma}_{xy} \propto \mathbf{v}_k$ that is cross-covariance and partial cross-covariance are equivalent (CCV ≡ PCCV). This equivalence can be extended to $\{x_t\}$ stationary autocorrelated, by assuming orthogonality between $[\psi_x(B)-1]$ and $v_b(B)$, hence

$$\gamma_{xy}(B) \cong v(B)B^b\sigma^2$$

Note that if $x_t \sim MA(q)$ with $q<b$, the above holds exactly because $((\psi_1, v_b)) \equiv 0$.

At multivariate level, given the relationship between partial covariance and model structure, the equivalence CCV ≡ PCCV follows from the possibility of reducing any AR model $\boldsymbol{\Phi}(B)\,\mathbf{z}_t = \mathbf{e}_t$ , $\boldsymbol{\Phi}(o) = \mathbb{I}$ , into a constrained TF-system

*(TFS*)*
$$\begin{cases} [\mathbb{I} - \tilde{\mathbf{W}}(B)]\,\mathbf{z}_t = \mathbf{n}_t , & \tilde{\mathbf{W}}(B)=\{\phi_{ij}(B)/\phi_{ii}(B)\} = \Sigma_{k=1}^{\infty} \tilde{\mathbf{W}}_k B^k \\ \mathbf{n}_t = \dot{\boldsymbol{\Phi}}(B)^{-1}\,\mathbf{e}_t , & \dot{\boldsymbol{\Phi}}(B) = \mathrm{Diag}[\phi_{ii}(B)] \end{cases}$$

Then by definition of partial correlations as marginal regression coefficients, from the first equation of the TFS* we have

$$E(\mathbf{z}_t\mathbf{z}'_{t-k} \mid \mathbf{z}_{t-1} \ldots \mathbf{z}_{t-k+1}) \propto \tilde{\mathbf{W}}_{kk} \neq \mathbb{O} \ , \ k>p$$

in the $i\neq j$ elements . In so doing an autoregression may exhibit both CCVF, PCCVF *infinite*, but this is possible only if the two are equivalent .

*Remark* - As said, the condition of polynomial orthogonality has important consequences in simplifying the identification. We recall that if

$$\begin{matrix} y_t \sim AR(p),\ p<b \\ x_t \sim MA(q),\ q<b \end{matrix} \quad \Rightarrow \quad \begin{matrix} \gamma_{nn}(B) = \gamma_{yy}(B) \\ \gamma_{xy}(B) = v(B)B^b\sigma^2 \end{matrix}$$

thus, $\psi(B)$, $v_b(B)$ are identifiable directly on the sample correlation functions avoiding filtering and prewhitening of sort .

## 2. IDENTIFICATION Of TFS

Let $\mathbb{z}'_t = [z_{1t} \dots z_{m_t}]$ , be a Gaussian stationary process mean-square summable

$$\{\mathbb{z}_t\} \sim N_m(\mathbb{o},\{\Gamma_k\}) \quad , \quad E(\mathbb{z}_t\mathbb{z}'_{t-k}) = \Gamma_k \qquad \Sigma_{k=o}^{\infty} \|\Gamma_k\| < \infty$$

and its transfer functions system (TFS) representation

(2.1) $$[\mathbb{I} - \mathbb{V}(B)]\, \mathbb{z}_t = \dot{\Psi}(B)\, \mathbb{a}_t \quad , \quad \mathbb{a}_t \sim IN_m(o, \dot{\Sigma}=\text{Diag})$$

$$\mathbb{V}(B) = \{\omega_{ij}(B)B^{b_{ij}}/\delta_{ij}(B)\} \quad ; \quad \dot{\Psi}(B) = \text{Diag}[\theta_i(B)/\phi_i(B)]$$

where $[\omega_{ij}(B),\delta_{ij}(B),\theta_i(B),\phi_i(B)]$ are linear polynomials of order $(s_{ij},r_{ij},q_i,p_i)<\infty$ , $b_{ij}$ is the delay factor of $z_{j_t}$ on $z_{i_t}$ and trace$\mathbb{V}(B)=0$ . In what follows the arguments are expounded in brief statements and informal demonstrations .

### *(2.1) Classical Properties*

The *classical* (overparametrized) MA-representation and CV-factorization are

(2.2a) $$\mathbb{z}_t = [\,\mathbb{I} - \mathbb{V}(B)]^{-1}\dot{\Psi}(B) = \Psi(B)\, \mathbb{a}_t$$

(2.2b) $$\Gamma(B) = \Psi(B)\dot{\Sigma}\, \Psi(F)'$$

*(Invertibility-Stationarity)* - Let $\mathbb{z}_t \sim$ TFS and $\text{Det}[\mathbb{I} - \mathbb{V}(B)] = \omega^*(B)/\delta^*(B)$ (say); then the classical conditions of *stability* are given by :

(2.3a) *invertibility* $[\delta_{ij}(B),\ \theta_i(B)] \neq 0 \quad , \quad |\omega_{ij}(B)| < \infty$

(2.3b) *stationarity* $[\delta_{ij}(B),\ \phi_i(B),\ \omega^*(B)] \neq 0$ *in* $|B| \leq 1$

Invertibility conditions are immediate. As regards (2.3b), given the relationship between stationarity and MA-decomponibility (multivariate Wold-Zashuin theorem, see Wiener-Masani(1957)p.137), from (2.2a) we write

$$[(\mathbb{I}-\mathbb{V}_o) - \Sigma_{k=1}^{\infty}\mathbb{V}_k B^k]\,[\Sigma_{k=o}^{\infty}\Psi_k B^k] = [\,\mathbb{I} + \Sigma_{k=1}^{\infty}\dot{\Psi}_k B^k]$$

Now, by equating products of matrices corresponding to the same powers of B, the recursive expression of the $\{\Psi_k\}$ sequence is

$$\Psi_k = (\,\mathbb{I} - \mathbb{V}_o)^{-1}\,[\Sigma_{j=1}^{k}\, \mathbb{V}_j\, \Psi_{k-j} - \dot{\Psi}_k]$$

that converges only if $\{\mathbb{V}_k\}$ converges, i.e. the $\delta_{ij}(B)$ are stable.

From another point of view, having

$$\Psi(B) = [\mathbb{I} - \mathbb{V}(B)]^* \, \dot{\Psi}(B) \; \delta^*(B)/\omega^*(B)$$

the adjoint matrix $[\mathbb{I}-\mathbb{V}(B)]^*$ has rational polynomials with stable denominators because they are formed by products of the $\delta_{ij}(B)$ ; the stability is then completed by the requirement on $\omega^*(B)$ .

*Remark* - Unlike the ARMA$_m$ analysis we may note that: i) invertibility and stationarity are *interdependent* since they require common conditions on the $\delta_{ij}(B)$; ii) the stability of $\omega^*(B)$, i.e. on the determinant, is not a *sufficient* condition of stationarity . Note also that although $\delta^*(B)=\prod_{i\neq j}\prod \delta_{ij}(B)$ (if the $\delta_{ij}(B)$ are prime) it is not possible to establish *disaggregate* conditions on the $\omega_{ij}(B)$ to ensure the stability of $\omega^*(B)$ .

*(Structural Identification)* - Let $\mathbb{z}_t \sim$ TFS ; then the factorization (2.2b) of the covariance functions matrix $\Gamma(B)=\{\gamma_{ij}(B)\}$ is uniquely identified if :
i) $[\omega_{ij}(B),\delta_{ij}(B)]$ $\forall$ ij, $[\theta_i(B),\phi_i(B)]$ $\forall$ i, are relatively prime by pair ;
ii) the stability conditions (2.3) hold ;
iii) $[\delta_{ij}(o) = \phi_i(o) = \theta_i(o) = 1$ , $\omega_{ij}(o) = \omega_{ji}(o)]$ ; i.e. $\mathbb{V}_o = \mathbb{V}_o'$ , $\dot{\Sigma}$ = Diag

Having $v_{ii}(B)=1$ $\forall$ i , the polynomials $[v_{i1}(B) \ldots v_{im}(B),\psi_i(B)]$ are relatively prime by row, hence, under the i)condition, the matrices $[\mathbb{I} - \mathbb{V}(B)]$ , $\dot{\Psi}(B)$ are left coprime. This means that their only admissible greatest common left divisor is a unimodular matrix $\mathbb{U}(B)$ (see Hannan(1969). Now since by definition $\mathbb{U}(B)$ is linear and Det $\mathbb{U}(B)$ is constant, the diagonality of $\dot{\Psi}(B)$ involves $\mathbb{U}(B) = \dot{\mathbb{U}}$ , constant and diagonal .

About the ii)condition, we note that any system matrix $\tilde{\Psi}(B)=\Psi(B)\dot{\mathbb{H}}(B)\mathbb{Q}$ , with $\dot{\mathbb{H}}(B)=\text{Diag}[h_i(B)/h_i(F)]$ and $\mathbb{Q}$ orthogonal, also satisfies the factorization of $\Gamma(B)$ because $\dot{\mathbb{H}}(B)\mathbb{Q}\mathbb{Q}'\dot{\mathbb{H}}(F)=\mathbb{I}$ . The matrices $\dot{\mathbb{H}}(B)$ however, cannot enjoy both (2.3a), (2.3b) ; indeed, if $h_i(B)$ has roots in $|B|>1$ , $h_i(F)$ must have roots in $|B|<1$ . The sole admissible $\dot{\mathbb{H}}(B)$ is then $\dot{\mathbb{U}}$, but the conditions in iii) restrict $\mathbb{Q} = \dot{\mathbb{U}} = \mathbb{I}$ uniquely .

Finally the specification $\mathbb{V}_o = \mathbb{V}_o'$ and/or $\Psi_o = \Psi_o'$, is identified because assuming $\Gamma_k = \mathbb{O}$ k≠o and $\dot{\Sigma} = \mathbb{I}$, we would have $\Gamma_o = \Psi_o \Psi_o'$ , thus $\Psi_o = \sqrt{\Gamma_o} = \mathbb{P}\sqrt{\Lambda}\,\mathbb{P}'$, which is positive definite and symmetrical .

*Remark* - Note that since by linearization a TFS corresponds to a canonical $ARMA_m$ form with $\Theta(B)$ diagonal the condition $\mathrm{Rank}[\Phi_p, \Theta_q] = m$, of Hannan(1969 , is not required here .

The realizability of the TFS-representation is ensured by the multivariate spectral factorization theorem of Rozanov(1967)p.47, extended by Hannan(1979).

*(Rational Realization) - Let* $\Gamma(e^{-i\omega})$, $[-\pi<\omega<\pi]$, *an m·m matrix, hermitian, positive definite, rational and integrable. Then an m·m matrix* $\Psi(z)$, *rational, non-singular, analytic in* $|z|\leq 1$ *, exists such that :* $\Gamma(e^{-i\omega}) = \Psi(e^{-i\omega})\,\Psi(e^{+i\omega})'$ .
*(The factorization is unique if* $\Psi(z)^{-1}$ *is analytic in* $|z| \leq 1$ *and* $\Psi(o)=\Psi(o)'$*)*

Looking at $\Gamma(e^{-i\omega})$ as the spectral density of $\{z_t\}$, and since by gaussianity $\{z_t\}$ is completely characterized by $\Gamma(z)$ , the theorem provides the basis for the existence and the uniqueness of the TFS-representation in the form (2.2a).

More precisely, since $\Psi(z)$ has the meaning of $\sqrt{\Gamma}(z)$, by the continuity of $\Gamma(z)$ in $z$ and $\{z_t\}$ definite in variance, we have $\lim_{z\to o}\sqrt{\Gamma}(z) = \sqrt{\Gamma_o} < \infty$, which implies that $\sqrt{\Gamma}(z)$ is holomorphic in a circular neighbourhood of $(z=o)$ (see Saks-Zygmund(1971)p.145). $\sqrt{\Gamma}(z)$ then admits a one-sided power expansion $\sqrt{\Gamma}(z) = \Sigma_{k=o}^{\infty} \Psi_k z^k$ , that by ergodicity of $\{z_t\}$ converges in $|z|\leq 1$ and so defines a function $\Psi(z)$ which is analytic there .

Otherwise, in order that $\sqrt{\Gamma}(z)$ be holomorphic in an annulus $\alpha<z<\alpha^{-1}$, $o<\alpha<1$, and so two-sided expansible there, the above limit $(\sqrt{\Gamma_o})$ should not exist (see Saks-Zygmund(1971)p.144). A condition clearly pathological for the process $\{z_t\}$ , but often implicitly assumed .

*Remark* - The Rozanov' theorem is usually used to maintain the realizability of a *general* $ARMA_m$ representation (see Hannan(1969, 1979)). In that context, however, the existence of the *further* factorization $\Psi(z) = \Phi(z)^{-1}\Theta(z)$ , with $\Phi(z)$ $\Theta(z)$ m·m linear and non-singular, should be required. Now, assuming $\Psi(z) = \{a_{ij}(z)/b_{ij}(z)\}$, a necessary condition becomes $b_{ij}(z)=|\Phi(z)|$ $\forall\, ij$ , which is not admissible if the $b_{ij}(z)$ are different or the $[a_{ij}(z), b_{ij}(z)]$ are prime .

### *(2.2) Realization of TFS*

Given the realization $\Pi(z) = \sqrt{\Gamma}(z)^{-1}$ and the associated rational AR-representation $(RAR_m)$ :

$$(RAR_m) \qquad \mathbb{\Pi}(B)\, \mathbb{z}_t = \mathbb{e}_t \quad , \quad \mathbb{e}_t \sim IN_m(\mathbb{o}, \Sigma>0)$$

$$\mathbb{\Pi}(B) = \left[\mathbb{I} - \Sigma_{k=1}^{\infty} \mathbb{\Pi}_k B^k\right] = \left[\mathbb{I} - \mathbb{\Pi}_1(B)\right]$$

to rise a TFS-structure , the rational matrix $\mathbb{\Pi}(z)$ must be factorizable as

$$\mathbb{\Pi}(z) = \left[\mathbb{I} - \mathbb{V}(z)\right] \; \dot{\mathbb{\Pi}}(z) \quad , \quad \dot{\mathbb{\Pi}}(z) = \dot{\Psi}(z)^{-1}$$

Now, letting $\mathbb{\Pi}(z) = [\dot{\mathbb{\Pi}}(z) - \mathbb{V}(z)]$ , where $\dot{\mathbb{\Pi}}(z)$ is the matrix formed with the principal diagonal of $\mathbb{\Pi}(z)$, the factorization may be proved with the arguments of the polynomial orthogonality, and more generally with the following results.

*(Orthogonal Projector)* - Let $\mathbb{z}_t \sim RAR_m$ , stable and identified; then $\mathbb{\Pi}_1(B)$ satisfying $\Sigma_{k=1}^{\infty} \|\mathbb{\Pi}_k\| < \infty$ , is idempotent and self-adjoint : $\mathbb{\Pi}_1(B) = \mathbb{\Pi}_1(F)' = \mathbb{\Pi}_1(B)^2$ .

*Geometrical Approach* - By definition an orthogonal projector is a linear operator that splits the space of definition in two orthogonal subspaces. Thus, let $H^-(t)$ be the Hilbert space formed by the closure in mean square convergence of the linear manifold generated by $\{\mathbb{z}_{t-k}\}$ , with inner product $((\mathbb{z}_t, \mathbb{z}_s)) = E(\bar{\mathbb{z}}_t' \, \mathbb{z}_s)$ ; and let $\mathcal{D}(t)$ be the orthogonal complement of $H^-(t-1)$ in $H^-(t)$.

Now, $\mathbb{\Pi}_1(B)$ is a linear transformation on $H^-(t)$ such that

$$\begin{aligned} \mathbb{\Pi}_1(B)\, \mathbb{z}_t &= E(\mathbb{z}_t \mid \mathbb{z}_{t-1}, \mathbb{z}_{t-2} \dots) = \hat{\mathbb{z}}_{t-1} \quad \varepsilon \; H^-(t-1) \\ [\mathbb{I} - \mathbb{\Pi}_1(B)]\, \mathbb{z}_t &= (\mathbb{z}_t - \hat{\mathbb{z}}_{t-1}) = \mathbb{e}_t \quad \varepsilon \; \mathcal{D}(t) \end{aligned} \qquad \forall\, \mathbb{z}_t \, \varepsilon\, H^-(t)$$

in fact $((\mathbb{e}_t, \mathbb{z}_{t-k})) = o$, $k>o$, and $\mathcal{D}(t)$ is generated by $\mathbb{e}_t$ itself . Therefore, $\mathbb{\Pi}_1(B)$ is the orthogonal projector of $H^-(t)$ *on* $R(\mathbb{\Pi}_1) = H^-(t-1)$ *along* $N(\mathbb{\Pi}_1) = \mathcal{D}(t)$.

The self-adjoint property follows because $\mathbb{\Pi}(z)^{-1} = \sqrt{\mathbb{F}}(z) = \mathbb{P}(z)\sqrt{\mathbb{\Lambda}}(z)\, \mathbb{P}(z^{-1})'$ which is hermitian. The idempotency of $\mathbb{\Pi}_1(B)$ (or that of $\mathbb{\Pi}(B)$), due to $H^-(t-1) \cap \mathcal{D}(t) = \emptyset$ (see Rao-Mitra(1971)p.109), follows by

$$((\mathbb{e}_t, \hat{\mathbb{z}}_{t-1})) = E\{\mathbb{z}_t' \,[\mathbb{I} - \mathbb{\Pi}_1(F)']\; \mathbb{\Pi}_1(B)\mathbb{z}_t\} = 0$$

It assumes the operative meaning

$$\mathbb{\Pi}(B)^2 \mathbb{z}_t = \mathbb{\Pi}(B)\, \mathbb{e}_t = \tilde{\mathbb{e}}_t \quad , \quad E(\tilde{\mathbb{e}}_t \, \tilde{\mathbb{e}}_{t-k}') = \mathbb{O}$$

that is allowed by the assumption that $\mathbb{\Pi}(B)$ is stable .

*Analytic Approach* - This approach relates to the properties of the analytic functions (see Saks-Zygmund(1971)pp.143-147). Suppose $\{\mathbb{z}_t\}$ to have a rational spectral density $\Gamma(z) = \{\gamma_{ij}(z)\}$ . By rationality each $\gamma_{ij}(z)$ belongs to the class of *meromorphic* functions, i.e. functions that cannot admit Laurent (two-sided) expansions on the entire closed plane $\not{C}$ . Thus in $z=0$ we can define only a Taylor (one-sided) expansion which converges to a function $\tilde{\Psi}(z)$ analytic

(2.4) $$\Gamma(z) = \Sigma_{k=o}^{\infty} \Gamma_k z^k \longrightarrow \tilde{\Psi}(z)$$

This holds for each $z \in \not{C}$ (by a shift of the origin), except at the poles $\hat{z}_{ij}$ of multiplicity $m_{ij}$ , where, however, we do not have a two-sided expansion

$$\Gamma(z) = \Sigma_{k=-m_{ij}}^{\infty} \Gamma_k (z - \hat{z}_{ij})^k$$

Now, from the multivariate spectral factorization theorem, if $\Gamma(z)$ is bounded and non-negative definite, we must have

(2.5) $$\Gamma(e^{-i\omega}) = \Psi(e^{-i\omega})\, \Psi(e^{+i\omega})' \longrightarrow \Sigma_{k=-\infty}^{\infty} \Gamma_k\, e^{-i\omega k}$$

in practice a two-sided expansion.

Thus, for $z = e^{-i\omega}$ the reconciliation of (2.4), (2.5) clearly requires

$$\Psi(z) = \Psi(z)^2 = \Psi(z^{-1})'$$

and the second order stationarity may be defined by construction as

$$\Gamma(z) = \tilde{\Psi}(z) + \tilde{\Psi}(z^{-1})' - \tilde{\Psi}(o)$$

*Algebraic Approach* - With conventional algebraic operators one may only find *quasi*-idempotency ; the approach is developed in the state-space context through the so-called "positive-real Lemma" (see Faurre *et al.*(1979)pp.26-126).

Linearizing the TFS we obtain a canonical $ARMA_m$ which may be cast in state-space form :

$$\begin{bmatrix} \mathbb{x}_{t+1} = \Phi\, \mathbb{x}_t + \Theta\, \mathbb{e}_t \\ \mathbb{z}_t = \mathbb{H}\, \mathbb{x}_t \end{bmatrix} , \qquad \Psi(z) = \mathbb{H} \left[ z\mathbb{I} - \Phi \right]^{-1} \Theta$$

Reasoning, for simplicity, in the continuous, we have

$$\begin{aligned} \Gamma(\omega) &= \Psi(-\omega)'\, \Psi(\omega) \\ &= \Theta' \left[ -\omega\, \mathbb{I} - \Phi' \right]^{-1} \mathbb{H}' \mathbb{H} \left[ \omega\, \mathbb{I} - \Phi \right]^{-1} \Theta \end{aligned}$$

Since $\Gamma(\omega)$ is non-negative definite, there exists a matrix $\mathbb{P} > 0$ , unique solution of the matricial equation

(2.6) $$\Phi'\mathbb{P} + \mathbb{P}\,\Phi = -\,\mathbb{H}'\mathbb{H} \geq 0$$

which takes on the meaning $\mathbb{P} = E(\mathbb{x}_t \mathbb{x}_t')$ . Now, letting

(2.7) $$\mathbb{L}' = \mathbb{P}\,\Theta$$

with some matrix algebra, from (2.6), (2.7) we obtain

$$\begin{aligned}\Gamma(\omega) &= \Theta'\left[-\omega\mathbb{I} - \Phi'\right]^{-1}\left\{\left[-\omega\mathbb{I} - \Phi'\right]\mathbb{P} + \mathbb{P}\left[\omega\mathbb{I} - \Phi\right]\right\}\left[\omega\mathbb{I} - \Phi\right]^{-1}\Theta \\ &= \mathbb{L}\left[\omega\mathbb{I} - \Phi\right]^{-1}\Theta + \Theta'\left[-\omega\mathbb{I} - \Phi'\right]^{-1}\mathbb{L}' \\ &= \tilde{\Psi}(\omega) + \tilde{\Psi}(-\omega)'\end{aligned}$$

Hence, the functions $\Psi(z)$, $\tilde{\Psi}(z)$ differ for the the *observation* matrices $\mathbb{L}$, $\mathbb{H}$ only, and have common system parameters $\Phi$, $\Theta$ .

*(TFS - Realization)* - Let $\mathbb{z}_t \sim RAR_m$ , stable and identified; then $\mathbb{II}(B)$ admits the factorization $\mathbb{II}(B) = [\mathbb{I} - \dot{\mathbb{II}}_1(B)]\,[\mathbb{I} - \mathbb{V}_1(B)]$, where $\dot{\mathbb{II}}_1(B) + \mathbb{V}_1(B) = \mathbb{II}_1(B)$. This factorization yields the decomposition $H^-(t-1) = A^-(t-1) \oplus C^-(t-1)$ : the subspaces of the Auto-correlated and Cross-correlated processes .

Since $\mathbb{II}_1(B)$ is a projector, by idempotency it follows that

$$\mathbb{II}_1(B) = \mathbb{II}_1(B)^2 \quad\Rightarrow\quad \dot{\mathbb{II}}_1(B) = \dot{\mathbb{II}}_1(B)^2,\ \mathbb{V}_1(B) = \mathbb{V}_1(B)^2,\ \dot{\mathbb{II}}_1(B)\mathbb{V}_1(B) = \mathbb{V}_1(B)\dot{\mathbb{II}}_1(B) = 0$$

thus, $\dot{\mathbb{II}}_1(B)$, $\mathbb{V}_1(B)$ are themselves projectors, and the factorizations

$$[\mathbb{I} - \mathbb{II}_1(B)] = [\mathbb{I} - \dot{\mathbb{II}}_1(B)][\mathbb{I} - \mathbb{V}_1(B)] = [\mathbb{I} - \mathbb{V}_1(B)][\mathbb{I} - \dot{\mathbb{II}}_1(B)]$$

hold. As a consequence any $RAR_m$ can be written in the sequential forms

*TFS(1)*
$$\begin{cases}[\mathbb{I} - \mathbb{V}_1(B)]\,\mathbb{z}_t = \mathbb{n}_t & \rightarrow \quad \mathbb{z}_t = \hat{\mathbb{z}}_{t-1}^{(C)} + \mathbb{n}_t \\ [\mathbb{I} - \dot{\mathbb{II}}_1(B)]\,\mathbb{n}_t = \mathbb{e}_t & \qquad\quad \hat{\mathbb{z}}_{t-1}^{(C)} \in C^-(t-1) \subset H^-(t-1)\end{cases}$$

*TFS(2)*
$$\begin{cases}[\mathbb{I} - \dot{\mathbb{II}}_1(B)]\,\mathbb{z}_t = \mathbb{u}_t & \rightarrow \quad \mathbb{z}_t = \hat{\mathbb{z}}_{t-1}^{(A)} + \mathbb{u}_t \\ [\mathbb{I} - \mathbb{V}_1(B)]\,\mathbb{u}_t = \mathbb{e}_t & \qquad\quad \hat{\mathbb{z}}_{t-1}^{(A)} \in A^-(t-1) \subset H^-(t-1)\end{cases}$$

where the first equations yield two type of projections on $H^-(t-1)$ and two types of uncorrelated processes. More precisely, if $N^-(t)$, $U^-(t)$ are the subspaces generated by $\{\mathbb{n}_{t-k}\}_0^\infty$, $\{\mathbb{u}_{t-k}\}_0^\infty$ then

$\mathbb{V}_1(B)$ *decomposes* $H^-(t) = C^-(t-1) \oplus N^-(t)$ , $N^-(t) = N(\mathbb{V}_1)$

$\dot{\mathbb{II}}_1(B)$ *decomposes* $H^-(t) = A^-(t-1) \oplus U^-(t)$ , $U^-(t) = N(\dot{\mathbb{II}}_1)$

Finally, since "*two projectors* $\dot{\mathbb{I}}_1$, $\mathbb{V}_1$ *such that* $\dot{\mathbb{I}}_1\mathbb{V}_1 = \mathbb{V}_1\dot{\mathbb{I}}_1 = 0$ *form a projector* $\mathbb{I}_1 = \dot{\mathbb{I}}_1 + \mathbb{V}_1$ *on* $R(\mathbb{I}_1) = R(\dot{\mathbb{I}}_1) \oplus R(\mathbb{V}_1)$ *along* $N(\mathbb{I}_1) = N(\dot{\mathbb{I}}_1) \cap N(\mathbb{V}_1)$" (see Rao Mitra(1971)p.107), it follows that

$$H^-(t-1) = A^-(t-1) \oplus C^-(t-1), \qquad \mathcal{D}(t) = U^-(t) \cap N^-(t)$$

$$H^-(t) = A^-(t-1) \oplus C^-(t-1) \oplus \mathcal{D}(t)$$

that is $\mathcal{D}(t)$ is a *splitting subspace* for $U^-(t)$, $N^-(t)$ in $H^-(t)$.

*Remark* - Although it is very sensible to filter $\{\mathbb{z}_t \to \mathbb{e}_t\}$ sequentially, starting with the univariate ARMA models $\dot{\mathbb{I}}(B)$ , from the works of Haugh-Box(1977),Granger -Newbold(1977)p.234, the representation TFS(2) would not be admissible because, by the covariance properties of the system,a process $\{\mathbb{u}_t\}$ with ACV=0, CCV≠0 could never be generated :

$$\Gamma_{\mathbb{u}}(B) = [\mathbb{I} - \mathbb{V}_1(B)]^{-1}\Sigma\, [\mathbb{I} - \mathbb{V}_1(F)']^{-1} \quad \Rightarrow \qquad \gamma_{u_i u_i}(B) \neq \sigma^2_{u_i}$$

i.e. ACV($\mathbf{u}$)≠0. A process like $\{\mathbb{u}_t\}$ might then be defined only by rewriting the second equation of TFS(2) as

$$[\mathbb{I} - \mathbb{V}_1(B)]\mathbb{u}_t = \dot{\Psi}(B)\mathbb{e}_t \quad \to \quad tr\{[\mathbb{I} - \mathbb{V}_1(B)]^{-1}\dot{\Psi}(B)\Sigma\dot{\Psi}(F)\, [\mathbb{I} - \mathbb{V}_1(F)']^{1}\} = c$$

which, however, contradicts the first equation and yields overparametrization .

In reality this situation might depend on the underlying MA-representation $\mathbb{u}_t = [\mathbb{I} - \mathbb{V}_1(B)]^{-1}\mathbb{e}_t$ (of explosive degree $\sum\sum_{i\neq j} r_{ij}$), and we may ask ourselves :
i) Has this *algebraic* decomposition sense from a stochastic point of view ?
ii) Does a *parsimonious* MA-representation exist, consistent with the TFS(2) ?
We try to answer in the next section. Now we consider the following example.

*Example* - Let $\{x_t, y_t\}$ be a zero mean process with covariances $E(x_t y_{t-h}) \propto \alpha$ and $E(y_t x_{t-k}) \propto -\beta$ . The TFS-representation and its MA-decomposition are then

$$\begin{bmatrix} 1 & -\alpha B^h \\ +\beta B^k & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} a_t \\ e_t \end{bmatrix} \quad , \quad \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & +\alpha B^h \\ -\beta B^k & 1 \end{bmatrix} \begin{bmatrix} a_t \\ e_t \end{bmatrix}$$

Indeed, multiplying on the left the first system by $\dot{\mathbb{Z}} = \mathrm{Diag}[y_{t-h}, x_{t-k}]$ and taking expectation, we obtain the assumed covariances; for the second system we have

$$E(x_t y_{t-h}) = E[(a_t + \alpha e_{t-h})(e_{t-h} - \beta a_{t-h-k})] \propto +\alpha$$
$$E(y_t x_{t-k}) = E[(e_t - \beta a_{t-k})(a_{t-k} + \alpha e_{t-k-h})] \propto -\beta$$

Owing to the equivalence CCV ≡ PCCV, this situation indicates that an "algebraic decomposition" cannot take place here .

*(2.3) Decomposition of TFS*

The questions asked above require that a non-algebraic solution be sought for.

*(TFS Decomposition)* - Let $\mathbb{z}_t \sim$ TFS(2) stable and identified ; then its inversion reduces to the rational moving average ($RMA_m$) structure :

(2.8) $$\begin{cases} \mathbb{z}_t = [\mathbb{I} + \dot{\Psi}_1(B)]\, \mathbb{u}_t \\ \mathbb{u}_t = [\mathbb{I} + \mathbb{V}_1(B)]\, \mathbf{e}_t \end{cases} \rightarrow \quad \mathbb{z}_t = [\mathbb{I} + \dot{\Psi}_1(B) + \mathbb{V}_1(B)]\, \mathbf{e}_t \qquad (RMA_m)$$

The first equation is immediate. As for the second, the proof may follow two approaches in which, treating the simultaneous causality later on, we assume $\Sigma = \dot{\Sigma}$.

*Deterministic Approach* - Since $\mathbb{V}_1(B)$ is a projector we have

$$(\Sigma_{k=1}^{\infty} \mathbb{V}_k B^k) = (\Sigma_{k=1}^{\infty} \mathbb{V}_k B^k)^2 \Rightarrow \mathbb{V}_i B^i \mathbb{V}_j B^j = \mathbb{V}_j B^j \mathbb{V}_i B^i = \begin{matrix} \mathbb{V}_i B^i & i=j \\ \mathbf{0} & i \neq j \end{matrix}$$

by which the factorization of the filter $[\mathbb{I} - \mathbb{V}_1(B)]$ easily follows

(2.9) $$[\mathbb{I} - \Sigma_{k=1}^{\infty} \mathbb{V}_k B^k] = \Pi_{k=1}^{\infty} [\mathbb{I} - \mathbb{V}_k B^k] = [\mathbb{I} - \mathbb{V}_k B^k]\ {}_{k=1}^{\infty}\Pi$$

The last expression enables to filter $\{\mathbb{u}_t \rightarrow \mathbf{e}_t\}$ sequentially as

$$\begin{array}{lll} (\mathbb{I} - \mathbb{V}_1 B^1)\, \mathbb{u}_t = \mathbb{u}_t^{(1)} & , \quad E(\mathbb{u}_t^{(1)} \mathbb{u}_{t-1}^{(1)\prime}) = \mathbb{0} & \\ (\mathbb{I} - \mathbb{V}_2 B^2)\, \mathbb{u}_t^{(1)} = \mathbb{u}_t^{(2)} & , \quad E(\mathbb{u}_t^{(2)} \mathbb{u}_{t-2}^{(2)\prime}) = \mathbb{0} & \\ \cdots\cdots\cdots & \quad \cdots\cdots\cdots & \\ (\mathbb{I} - \mathbb{V}_k B^k)\, \mathbb{u}_t^{(k-1)} = \mathbb{u}_t^{(k)} & , \quad E(\mathbb{u}_t^{(k)} \mathbb{u}_{t-j}^{(k)\prime}) = \mathbb{0} & j \leq k \end{array}$$

$$\{\mathbb{u}_t^{(k)}\} \xrightarrow[k \rightarrow \infty]{ms} \{\mathbf{e}_t\}$$

Now, under stability, the second equation of the TFS(2) can be inverted as

$$\mathbb{u}_t = [\mathbb{I} + \mathbb{W}_1(B)]\, \mathbb{e}_t \quad , \quad \mathbb{W}_1(B) \textit{ analytic in } |B| \leq 1$$

where, since $[\mathbb{I} + \mathbb{W}_1(B)]$ is the inverse of a projector it is idempotent. Hence

(2.10) $$[\mathbb{I} + \Sigma_{k=1}^{\infty} \mathbb{W}_k B^k] = [\mathbb{I} + \mathbb{W}_k B^k]\ {}_{k=1}^{\infty}\Pi$$

and $$(\mathbb{I} + \mathbb{W}_k B^k)\, \mathbb{e}_t^{(k-1)} = \mathbb{e}_t^{(k)} \quad , \quad E(\mathbb{e}_{t+k}^{(k)} \mathbb{e}_t^{(k)\prime}) \neq \mathbb{0} \quad j \leq k$$

Finally, having $[\mathbb{I} + \mathbb{W}_1(B)][\mathbb{I} - \mathbb{V}_1(B)] = \mathbb{I}$, from (2.9),(2.10) we obtain

$$[\mathbb{I} + \mathbb{W}_k B^k][\mathbb{I} - \mathbb{V}_k B^k] = \mathbb{I} \qquad \Rightarrow \qquad \mathbb{W}_k = +\mathbb{V}_K \qquad \forall\, k$$

by equating products of matrices corresponding to same powers of B .

*Stochastic Approach* - Under the general stationary condition $c\mathbb{I} < \Gamma(z) < \mathbb{I}c^{-1}$ $0 < c < 1$ (that is, if the spectral density of $\{\mathbb{z}_t\}$ is positive definite and bounded everywhere); Wiener-Masani(1958)p.119 showed that $H^-(t-1) = \Sigma_{k=1}^{\infty} \oplus H(t-k)$, with $H(t-k)$ the subspace of dimension one generated by $(\mathbb{z}_{t-k})$. If moreover $\{\mathbb{z}_t\}$ is purely non-deterministic, i.e. $H(-\infty) = \emptyset$, it is well known that $H^-(t-1) = \Sigma_{k=1}^{\infty} \oplus \mathcal{D}(t-k)$, with $\mathcal{D}(t-k)$ of dimension one and generated by $\mathbb{e}_{t-k} = \mathbb{z}_{t-k} - \hat{\mathbb{z}}_{t-k-1}$. The question that consequently arises is : When may we have $H(t-k) \equiv \mathcal{D}(t-k)$, i. e. when are the bases $\{\mathbb{z}_{t-k}\}_1^{\infty}$, $\{\mathbb{e}_{t-k}\}_1^{\infty}$ *exchangeable* in $H^-(t-1)$ ?

In the following treatment this seems to be the case whenever $\{\mathbb{z}_t\} \equiv \{\mathbb{u}_t\}$, for the general reason that for whitened series the equivalence CCV ≡ PCCV holds .

In the product $\Pi_k\, [\mathbb{I} - \mathbb{V}_k B^k]$ the arrangement of the linear factors may be anyhow, and we can isolate the k-th *CCV-state* of $\{\mathbb{u}_t\}$ as

$$\Pi_{j \neq k} [\mathbb{I} - \mathbb{V}_j B^j]\, \mathbb{u}_t = \mathbb{u}_t^{((k))}$$

$$\mathbb{u}_t^{((k))} = \mathbb{V}_k\, \mathbb{u}_{t-k}^{((k))} + \mathbb{e}_t \qquad\qquad E(\mathbb{u}_t^{((k))} \mathbb{u}_{t-j}^{((k))\,\prime}) = \mathbb{O} \qquad j \neq k$$

Multiplying the above on the right by $\mathbb{u}_{t-k}^{((k))\,\prime}$, and taking expectation we find

$$E(\mathbb{u}_t^{((k))} \mathbb{u}_{t-k}^{((k))\,\prime}) = \mathbb{V}_k\, \dot{\Sigma}_{\mathbb{u}} \qquad \to \qquad \Gamma_{\mathbb{u}}(k) \propto \mathbb{V}_k$$

The MA - representation of $\{\mathbb{u}_t^{((k))}\}$ consequently must be

$$\mathbb{u}_t = [\mathbb{I} + \mathbb{V}_k B^k]\, \mathbb{e}_t$$

since by substituting it in the expectation we satisfy

$$E[(\mathbb{e}_t + \mathbb{V}_k \mathbb{e}_{t-k})(\mathbb{e}_{t-k} + \mathbb{V}_k \mathbb{e}_{t-2k})'] = \mathbb{V}_k\, \dot{\Sigma}_{\mathbb{e}} \qquad \to \qquad \Gamma_{\mathbb{u}}(k) \propto \mathbb{V}_k$$

Finally, since the above holds for each k and for the whitened series CCV ≡ PCCV, we have

$$\Gamma_{\mathbb{u}}(B) = [\Sigma_{k=0}^{\infty}\, \Gamma_{\mathbb{u}}(k) B^k] \propto [\Sigma_{k=0}^{\infty}\, \mathbb{V}_k B^k] = [\mathbb{I} + \mathbb{V}_1(B)]$$

A result that agrees with the covariance properties of (2.8) since from it

$$\Gamma_{\mathbb{u}}(B) = [\mathbb{I} + \mathbb{V}_1(B)]\dot{\Sigma}\, [\mathbb{I} + \mathbb{V}_1'(F)] = [\mathbb{I} + 3\mathbb{V}_1(B)]\dot{\Sigma} \propto [\mathbb{I} + \mathbb{V}_1(B)]$$

because $\mathbb{V}_1(B)$ is idempotent, self-adjoint and $\dot{\Sigma}$ is diagonal.

From another point of view, the *exchangeability* of the bases $\{\mathbf{u}_{t-k}\}$, $\{\mathbf{e}_{t-k}\}$ is admissible if they have the same cov-relationships with $(\mathbf{u}_t)$ : $E(\mathbf{u}_t \mathbf{u}'_{t-k}) = E(\mathbf{u}_t \mathbf{e}'_{t-k})$, i.e. $\Gamma_{\mathbf{u}}(k) \propto \mathbb{W}_k$ . Now, having $\Gamma_{\mathbf{u}}(k) \propto \mathbb{V}_k$ , the condition for exchanging becomes $\mathbb{W}_k = \mathbb{V}_k$ .

To complete the proof of (2.8), from the orthogonality of $\dot{\mathbb{I}}_1(B)$ and $\mathbb{V}_1(B)$, it easily follows the orthogonality of $\dot{\Psi}_1(B)$ and $\mathbb{V}_1(B)$, in fact

$$\dot{\mathbb{I}}(B)\ \mathbb{V}_1(B) = \mathbb{V}_1(B) = \dot{\Psi}(B)\ \mathbb{V}_1(B)$$

*Remarks* - These results seem acceptable also in view of the following notes :

i) under stability we may write, at least approximately, $[\mathbb{I}-\Sigma_k \mathbb{V}_k B^k] \cong \Pi_k [\mathbb{I}-\mathbb{V}_k B^k]$ in any case these filters tend to have the same statistical performance ;

ii) the $RMA_m$ representation $\mathbf{z}_t = [\dot{\Psi}(B)+\mathbb{V}(B)]\mathbf{e}_t$ is parametrically equivalent to its corresponding $RAR_m$ $[\dot{\mathbb{I}}(B)-\mathbb{V}(B)]\mathbf{z}_t = \mathbf{e}_t$ . This property is consistent with the univariate analysis and implies that the two models are estimable with the same pseudolinear regression algorithm .

iii) The requirement $tr\,\Gamma_{\mathbf{u}}(B) =$ constant, that is $ACV(\mathbf{u})=0$, is satisfied, where, in fact, $\{\mathbf{u}_t\}$ may be obtained in the first step by filtering $\{\mathbf{z}_t\}$ with the univariate ARMA models of the individul series $\{z_{i_t}\}$ $i = 1,2 \ldots m$ .

iv) Consistently with the interdependence between stationarity and invertibility in the TFS , the general conditions of stability reduce to

$$[\delta_{ij}(B), \phi_i(B), \theta_i(B)] \neq 0\ , \quad |\omega_{ij}(B)| < \infty \quad \text{in} \quad |B| \leq 1 \quad \forall\, ij$$

Note that since these are completely disaggregate, it becomes possible to identify and control the specific linear factors, or roots, that cause instability in the system .

*(TFS Factorization)* - Let $\mathbf{z}_t \sim TFS(1)$ stable and identified ; then the univariate-unidirectional expressions of the covariance functions hold :

$$\gamma_{ij}(B) \propto v_{ij}(B)\gamma_{ii}(B)\ , \qquad \gamma_{ii}(B) \propto \psi_i(B)\psi_i(F)$$

Evidence of the result arise in two context. From the previous theory we have

$$\Gamma(B) = [\mathbb{I}+\dot{\Psi}_1(B)+\mathbb{V}_1(B)]\dot{\Sigma}[\mathbb{I}+\dot{\Psi}_1(F)+\mathbb{V}'_1(F)]$$

$$= [\mathbb{I}+\dot{\Psi}_1(B)+\mathbb{V}_1(B)]\dot{\Sigma}[\mathbb{I}+\mathbb{V}'_1(F)][\mathbb{I}+\dot{\Psi}_1(F)]$$

$$= \left[ \mathbb{I}+\dot{\Psi}_1(B)+ 3\mathbb{V}_1(B)\right] \dot{\Sigma} \left[ \mathbb{I}+\dot{\Psi}_1(F)\right]$$
$$= \left[\mathbb{I}+ 3\mathbb{V}_1(B)\right]\dot{\Sigma} \left[\mathbb{I}+\dot{\Psi}_1(B)\right] \left[ \mathbb{I}+\dot{\Psi}_1(F)\right]$$
$$\Gamma(B) \propto \left[ \mathbb{I} +\mathbb{V}_1(B)\right] \dot{\Psi}(B) \dot{\Psi}(F)$$

In practical terms, treating the TFS(1) by rows, from $[\mathbb{I} -\mathbb{V}(B)] = \mathbf{n}_t$ as in Box-Jenkins(1970)p.415, we obtain the equations

$$\begin{bmatrix} \gamma_{i1}(B) \\ \gamma_{i2}(B) \\ \vdots \\ \gamma_{im}(B) \end{bmatrix} = \begin{bmatrix} \gamma_{11}(B) & \gamma_{12}(B) & \dots & \gamma_{1m}(B) \\ \gamma_{21}(B) & \dots & \dots & \vdots \\ \vdots & & & \\ \gamma_{m1}(B) & \dots & \dots & \gamma_{mm}(B) \end{bmatrix} \begin{bmatrix} v_{i1}(B) \\ v_{i2}(B) \\ \vdots \\ v_{im}(B) \end{bmatrix}$$

i.e.
$$\boldsymbol{\gamma}_i(B) = \Gamma_{ii}(B) \ \mathbf{v}_i(B) \qquad i = 1,2 \dots m$$

Now, solving for $\mathbf{v}_i(B)$ by applying *stochastic approximation* arguments to the linear regression (see Tsypkin(1971)p.65), we may diagonalize $\Gamma_{ii}(B)$, obtaining $\gamma_{ij}(B)\cong\gamma_{ii}(B)v_{ij}(B)$. The second expression follows by $\mathbf{n}_t = \dot{\Psi}(B)\, \mathbf{e}_t$ assuming $ACV(\mathbf{m}) \cong ACV(\mathbf{z})$ .

*Remark* - These expressions can *at least* be accepted as approximations of the *true* algebraic ones, and used in the early phase of identification in which only a general idea of the system is required. Here, they enable the complex dynamic relationships inside a TFS to be disaggregated and the Box-Jenkins schemes to be applied . Good results can clearly be gained for moderately correlated time series ; in this case prewhitenig procedures may also be avoided .

The extension of the analysis in presence of simultaneous correlation is straightforward. Suppose $\Sigma > 0$, by normalizing and developing we have

$$\sqrt{\Sigma}^{-1} \left[\mathbb{I}- \mathbb{II}_1(B)\right]\mathbf{z}_t= \sqrt{\Sigma}^{-1}\mathbf{e}_t$$
$$(\mathbb{I} -\mathbb{V}_0) \left[\mathbb{I} -\mathbb{V}_1(B)\right] \left[ \mathbb{I} - \dot{\mathbb{II}}_1(B)\right] \mathbf{z}_t = \mathbf{a}_t$$

from which
$$(\mathbb{I} -\mathbb{V}_0) \left[\mathbb{I}-\Sigma_{k=1}^{\infty}\mathbb{V}_k B^k\right] \mathbf{u}_t = (\mathbb{I} -\mathbb{V}_0)\Pi_{k=1}^{\infty}\left[\mathbb{I} -\mathbb{V}_k B^k\right] \mathbf{u}_t = \mathbf{a}_t$$
$$\left[\mathbb{I} - \Sigma_{k=o}^{\infty}\mathbb{V}_k B^k\right] + (\Sigma_{k=1}^{\infty}\tilde{\mathbb{V}}_k B^k) = \Pi_{k=o}^{\infty}\left[ \mathbb{I} -\mathbb{V}_k B^k\right]$$

Now, by the formal relationships between *multiplicative* and *additive* forms the projectors, uniqueness of the projections on the same past events, structural properties of the cross projectors ($tr\ \mathbb{V}(B) = 0$), and since $\mathbb{V}_0 \in \mathcal{D}(t)$, $\mathbb{V}_1(B) \in H^-(t-1)$, we must have $\mathbb{V}_0\mathbb{V}_k= \tilde{\mathbb{V}}_k = \mathbb{O}\ \forall k$ . Thus, $(\mathbb{I}-\mathbb{V}_0)$ is a projector.

## 3. ESTIMATION Of TFS

The paper ends with an empirical comparison of the statistical performance of TFS and $ARMA_m$ models, applied to 5 real time series. In the present situation computer packages for the joint-estimation of simultaneous transfer functions equations are not available ; moreover, existing packages for the $ARMA_m$ have serious problems of convergence, initial values, and in many cases, when $m > 2$, they are not able to estimate even $AR_m$ models .

For both the models we now suggest approximate methods of estimation, easily implementable on Fortran programs. These algorithms, multivariate and iterative extensions of the recursive pseudolinear regression (R-PLR, see Solo (1981), Ljung-Söderström(1983), for the ARMAX model), are asymptotically efficient and, under the assumption of *passivity* for the monic polynomials of the system, they are strongly consistent .

### *(3.1) Pseudolinear Estimation of TFS*

Given the univariate-orthogonal structure of the residuals, the TFS may be initially estimated, without loss of efficiency and consistency, by rows, through non-linear least squares technics (NLS,see Box-Jenkins(1970)p.391).

For the i-th equation, assuming common orders $p,q,r,s,b$, we have

*(I-NLS)*

$$\hat{\beta}_i(k+1) = \hat{\beta}_i(k) + \left[\Sigma_{t=1}^{n} \hat{\xi}_{i_t}(k)\hat{\xi}'_{i_t}(k)\right]^{-1} \Sigma_{t=1}^{n} \hat{\xi}_{i_t}(k)\hat{a}_{i_t}(k)$$

$$\xi_{i_t}(\beta) = \partial a_{i_t}(\beta)/\partial \beta \quad , \quad a_{i_t}(\beta) = \pi_i(B)\left[z_{i_t} - \Sigma_{j\neq i}^{m} v_{ij}(B) z_{j_{t-b}}\right]$$

$$\beta'_i = [\delta_{i_1}(1)\ldots\delta_{i_1}(r), -\omega_{i_1}(0)\ldots\omega_{i_1}(s),\ldots\ldots\omega_{i_m}(s);\phi_i(1)\ldots\phi_i(p)\ldots\ldots\theta_i(q)]$$

To derive a useful expression fo the gradient we define the auxiliary variables

$$w_{ij_t} = [\omega_{ij}(B)/\delta_{ij}(B)]\ z_{j_{t-b}} \quad , \quad n_{i_t} = [\theta_i(B)/\phi_i(B)]\ a_{i_t}$$

now, it is not difficult to show that

$$\xi_{i_t}(\beta) \begin{cases} \partial a_{i_t}(\beta)/\partial\delta_{ij}(h) = -[\pi_i(B)/\delta_{ij}(B)]\ w_{ij_{t-h}} & h=1,2\ \ldots\ r \\ \partial a_{i_t}(\beta)/\partial\omega_{ij}(h) = \ \ [\pi_i(B)/\delta_{ij}(B)]\ z_{j_{t-b-h}} & h=0,1\ \ldots\ s \\ \partial a_{i_t}(\beta)/\partial\phi_i(h) = -[1/\theta_i(B)]\ n_{i_{t-h}} & h=1,2\ \ldots\ p \\ \partial a_{i_t}(\beta)/\partial\theta_i(h) = \ \ [1/\theta_i(B)]\ a_{i_{t-h}} & h=1,2\ \ldots\ q \end{cases}$$

The computation of the gradient thus consists in a filtering operation on observable, auxiliary and non-observable quantities. Note that in the case of polynomial orthogonality $\pi_i(B)/\delta_{ij}(B) \cong 1/\delta_{ij}(B)$, the calculation of the first two derivatives simplifies considerably.

The computation of the residual $\{a_{i_t}\}$ is carried out in 3 steps

1) $w_{ij_t} = \Sigma_{h=1}^{r} \delta_{ij}(h) w_{ij_{t-h}} - \Sigma_{h=0}^{s} \omega_{ij}(h) z_{j_{t-b-h}} \qquad j=(1,2\ ..\ m)\neq i$

2) $n_{i_t} = z_{i_t} - \Sigma_{j\ i}^{m} w_{ij_t}$

3) $a_{i_t} = n_{i_t} - \Sigma_{h=1}^{p} \phi_i(h) n_{i_{t-h}} + \Sigma_{h=1}^{q} \theta_i(h) a_{i_{t-h}}$

Recomposing the 3 steps, we may rewrite the model in pseudolinear form as

$$z_{i_t} = \Sigma_{j\neq i}^{m} (\boldsymbol{\delta}'_{ij} \mathbf{w}_{ij_{t-1}} - \boldsymbol{\omega}'_{ij} \mathbf{z}_{j_{t-b}}) + (\boldsymbol{\phi}'_i \mathbf{m}_{i_{t-1}} - \boldsymbol{\theta}'_i \mathbf{a}_{i_{t-1}}) + a_{i_t}$$

$$\mathbf{y}_{i_t}(\boldsymbol{\beta}) \left\langle \begin{array}{ll} \mathbf{w}_{ij_{t-1}} = [w_{ij_{t-1}} \ \dots \ w_{ij_{t-r}}]', & [\delta_{ij}(1) \ \dots \ \delta_{ij}(r)]' = \boldsymbol{\delta}_{ij} \\ \mathbf{z}_{j_{t-b}} = [z_{j_{t-b}} \ \dots \ z_{j_{t-b-s}}]', & [-\omega_{ij}(0) \ \dots \ \omega_{ij}(s)]' = \boldsymbol{\omega}_{ij} \\ \mathbf{m}_{i_{t-1}} = [n_{i_{t-1}} \ \dots \ n_{i_{t-p}}]', & [\phi_i(1) \ \dots \ \phi_i(p)]' = \boldsymbol{\phi}_i \\ \mathbf{a}_{i_{t-1}} = [a_{i_{t-1}} \ \dots \ a_{i_{t-q}}]', & [\theta_i(1) \ \dots \ \theta_i(q)]' = \boldsymbol{\theta}_i \end{array} \right\rangle \boldsymbol{\beta}_i$$

(3.1) $$z_{i_t} = \boldsymbol{\beta}'_i \, \mathbf{y}_{i_t}(\boldsymbol{\beta}) + a_{i_t}(\boldsymbol{\beta})$$

where $\mathbf{y}_{i_t}(\boldsymbol{\beta})$ is the vector of pseudolinear regressors .

The pseudolinear estimator arises from the non-linear one by approximating

(3.2) $$\hat{\boldsymbol{\varepsilon}}_{i_t}(k) \cong \hat{\mathbf{y}}_{i_t}(k)$$

i.e. by avoiding the filtering with $\pi_i(B)/\delta_{ij}(B)$ and $1/\theta_i(B)$. Moreover since

$$\hat{a}_{i_t}(k) = z_{i_t} - \hat{\boldsymbol{\beta}}'_i(k) \, \hat{\mathbf{y}}_{i_t}(k)$$

the iterative pseudolinear regression (I-PLR) algorithm reduces to

*(I-PLR)* $$\hat{\boldsymbol{\beta}}_i(k+1) = [\Sigma_{t=1}^{n} \hat{\mathbf{y}}_{i_t}(k) \hat{\mathbf{y}}'_{i_t}(k)]^{-1} \Sigma_{t=1}^{n} \hat{\mathbf{y}}_{i_t}(k) \, z_{i_t}$$

This algorithm also arises as OLS-estimator in the pseudolinear model (3.1), the substantial step is however the approximation (3.2). It is the goodness of this approximation that influences the statistical properties of the I-PLR. Generalizing the analysis developed by Ljung-Söderström(1983)Chap.4, for the recursive estimation of ARMAX models, we may conclude that the algorithm is strongly consistent if the monic polynomials of the system are strictly passive :

$$|\phi_i(z)/\theta_i(z)\delta_{ij}(z) - \tfrac{1}{2}| > 0 \quad , \quad |z| \leq 1 \quad \forall\, ij$$

note that only for first order polynomials do these coincides with the stability.

The philosophy of pseudolinearity may also be used to get good initial values $\hat{\boldsymbol{\beta}}_i(o)$ : by means of a set of linear regression we obtain the estimates

$$(3.3)\quad \begin{array}{ll} a) & z_{i_t} = \boldsymbol{\delta}'_{ij}\mathbf{z}^{(1)}_{i_{t-1}} - \boldsymbol{\omega}'_{ij}\mathbf{z}_{ij_{t-b}} + n_{i_t} \rightarrow \hat{\boldsymbol{\delta}}_{ij}(0),\ \hat{\boldsymbol{\omega}}_{ij}(0) \\ b) & \left[ \begin{array}{l} z_{i_t} = \boldsymbol{\alpha}'_i\mathbf{z}^{(2)}_{i_{t-1}} + e_{i_t} \rightarrow \hat{e}_{i_t}(0) \\ z_{i_t} = \boldsymbol{\phi}'_i\mathbf{z}^{(3)}_{i_{t-1}} - \boldsymbol{\theta}'_i\hat{\mathbf{e}}_{i_{t-1}} + u_{i_t} \rightarrow \hat{\boldsymbol{\phi}}_i(0),\ \hat{\boldsymbol{\theta}}_i(0) \end{array} \right. \end{array}$$

In the second step we first generete a white noise process $\{e_{i_t}\}$, through an autoregression of order $g > (p+q)$ ; the ARMA parameters are then estimated with a pseudolinear regression.

A *second* I-PLR algorithm arises by writing the system in simplified AR-form by means of the polynomial orthogonal approximation :

i-th row

$$\pi_i(B)z_{i_t} = \Sigma^m_{j\neq i}\ v_{ij}(B)\ z_{j_{t-b}} + a_{i_t}$$
$$u_{i_t} = \Sigma^m_{j\neq i}\ w_{ij_t} + a_{i_t}$$

From the auxiliary variable $u_{i_t} = \pi_i(B)z_{i_t}$ (which corresponds to an ARMA residual) the computation of $\{a_{i_t}\}$ again follows a 3 steps procedures with the first step equal to the previous one

$$2')\quad u_{i_t} = z_{i_t} - \Sigma^p_{h=1}\ \phi_i(h)z_{i_{t-h}} + \Sigma^q_{h=1}\ \theta_i(h)u_{i_{t-h}}$$
$$= z_{i_t} - \boldsymbol{\phi}'_i\ \mathbf{z}_{i_{t-1}} + \boldsymbol{\theta}'_i\ \mathbf{u}_{i_{t-1}}$$
$$3')\quad a_{i_t} = u_{i_t} - \Sigma^m_{j\neq i}\ w_{ij_t}$$

Recomposing the 3 steps the associated second pseudo-linear equation is

$$z_{i_t} = \Sigma^m_{j\neq i}(\boldsymbol{\delta}'_{ij}\mathbf{w}_{ij_{t-1}} - \boldsymbol{\omega}'_{ij}\mathbf{z}_{ij_{t-b}}) + (\boldsymbol{\phi}'_i\mathbf{z}_{i_{t-1}} - \boldsymbol{\theta}'_i\mathbf{u}_{i_{t-1}}) + a_{i_t}$$
$$= \boldsymbol{\beta}'_i\ \tilde{\mathbf{y}}_{i_t}(\boldsymbol{\beta}) + a_{i_t}(\boldsymbol{\beta})$$

Since the gradient in this form takes the structure

$$\xi_{i_t}(\boldsymbol{\beta}) \left\langle \begin{array}{l} \partial a_{i_t}(\boldsymbol{\beta})/\partial\delta_{ij}(h) = -[1/\delta_{ij}(B)]\ w_{ij_{t-h}} \\ \partial a_{i_t}(\boldsymbol{\beta})/\partial\omega_{ij}(h) = [1/\delta_{ij}(B)]\ z_{j_{t-b-h}} \\ \partial a_{i_t}(\boldsymbol{\beta})/\partial\phi_i(h) = -[1/\theta_i(B)]\ z_{i_{t-h}} \\ \partial a_{i_t}(\boldsymbol{\beta})/\partial\theta_i(h) = [1/\theta_i(B)]\ u_{i_{t-h}} \end{array} \right.$$

a second pseudolinear estimator (and with minor assumptions) is realizable .

Unlike the former, the two subvectors of regressors

$$[\mathbf{w}'_{ij_{t-1}}, \mathbf{z}'_{ij_{t-b}}] \quad , \quad [\mathbf{z}'_{i_{t-1}}, \mathbf{u}'_{i_{t-1}}] \qquad \forall\ j$$

are not stochastically independent, so the estimates $\hat{v}(B),\hat{\psi}(B)$ are not asymptotically independent. From a computational point of view, however, the subvectors contain only two pseudolinear quantities $\{w_{ij_t}, u_{i_t}\}$ which are filtered independently from the observable processes $\{z_{j_t}, z_{i_t}\}$ .

In presence of simultaneous causality and assuming the specification $\mathbb{V}_0 = \mathbb{O}$, i. e. $\omega_{ij}(0)=0\ \forall\ ij$, we may define an efficient system-estimator through a seemingly-unrelated structure as

$$\begin{aligned}
&\boldsymbol{\beta}' = [\boldsymbol{\beta}'_1\ \boldsymbol{\beta}'_2 \dots \boldsymbol{\beta}'_m], \qquad \boldsymbol{\beta}'_i = [\boldsymbol{\delta}'_{i1}\ \boldsymbol{\omega}'_{i1} \dots\dots \boldsymbol{\omega}'_{im}\ \boldsymbol{\phi}'_i\ \boldsymbol{\theta}'_i] \\
&\mathbb{W} = \mathrm{Diag}[\mathbb{Y}_1\ \mathbb{Y}_2 \dots \mathbb{Y}_m] \ , \qquad \mathbb{Y}'_i = [\mathbf{y}_{i_1}\ \mathbf{y}_{i_2} \dots \mathbf{y}_{i_n}] \\
&\mathbf{z}' = [\mathbf{z}'_1\ \mathbf{z}'_2 \dots \mathbf{z}'_m] \ , \qquad \mathbf{z}'_i = [z_{i_1}\ z_{i_2} \dots z_{i_n}]
\end{aligned}$$

$$\hat{\boldsymbol{\beta}}(k+1) = \{\hat{\mathbb{W}}(k)'[\hat{\Sigma}(k)\otimes\mathbb{I}]^{-1}\hat{\mathbb{W}}(k)\}^{-1}\hat{\mathbb{W}}(k)'[\hat{\Sigma}(k)\otimes\mathbb{I}]^{-1}\mathbf{z}$$

## *(3.2) Pseudolinear Estimation of ARMA$_m$*

The pseudolinear estimation of ARMA$_m$ models, has already been considered by Spliid(1983). His work however, does not provide a realization framework for the algorithm and his analysis of the statistical properties seems incomplete.

Since an ARMA$_m$(p,q) can be recast in a canonical ARMA$_M$(1,1) form, with $M=[m\cdot \max(p,q)]$, we consider, without loss of generality, the estimation of

$$\mathbf{z}_t = \boldsymbol{\Phi}\ \mathbf{z}_{t-1} - \boldsymbol{\Theta}\ \mathbf{e}_{t-1} + \mathbf{e}_t$$

Let $\boldsymbol{\beta} = \mathrm{Vec}[\boldsymbol{\Phi},\boldsymbol{\Theta}]$ and define the multivariate expansion

$$\begin{aligned}
&\mathbf{e}_t(\hat{\boldsymbol{\beta}}) \cong \Xi'_t(\hat{\boldsymbol{\beta}})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}) + \mathbf{e}_t(\boldsymbol{\beta}) \\
&\Xi'_t(\boldsymbol{\beta}) = [\partial\mathbf{e}_t(\boldsymbol{\beta})/\partial\boldsymbol{\beta}' = \{\partial e_{i_t}(\boldsymbol{\beta})/\partial\beta_j\}]_{m\cdot 2m^2}
\end{aligned}$$

the corresponding iterative non-linear least squares estimator is

$$\hat{\boldsymbol{\beta}}(k+1) = \hat{\boldsymbol{\beta}}(k) + [\Sigma_{t=1}^n \hat{\Xi}_t(k)\hat{\Xi}'_t(k)]^{-1}\ \Sigma_{t=1}^n \hat{\Xi}_t(k)\ \hat{\mathbf{e}}_t(k)$$

Now, using $\mathbf{e}_t = \boldsymbol{\Theta}(B)^{-1}\boldsymbol{\Phi}(B)\ \mathbf{z}_t$, a generic row of $\Xi_t$ is given by

$$\xi_{\ell_t}(\beta) \begin{cases} \partial \mathbf{e}_t(\beta)/\partial\phi_{ij} = -\Theta(B)^{-1}\, \mathbb{J}_{ij}\, \mathbf{z}_{t-1} \cong \mathbf{z}_{t-1} \\ \partial \mathbf{e}_t(\beta)/\partial\theta_{ij} = \Theta(B)^{-1}\, \mathbb{J}_{ij}\, \mathbf{e}_{t-1} \cong \mathbf{e}_{t-1} \end{cases}$$

where $\mathbb{J}_{ij}$ is a matrix with 1 in the ij-position and 0 elsewhere. As before, we may resonably approximate the gradient with input-output processes. For an $MA_m(1)$ model, however, we would have $\Xi'_t \cong [\mathbf{e}_{t-1}, \mathbf{e}_{t-1} \ldots \mathbf{e}_{t-1}] = \mathbb{E}'_t$ , and the matrix of squared regressors $[\Sigma_t \mathbb{E}_t \mathbb{E}'_t]$ , to be inverted in the estimator, is singular.

We may solve the problem by considering another form of expansion that works directly on the approximate gradient (pseudolinear regressors). It is

$$\mathbf{e}'_t(\hat{\mathbb{B}}) \cong \mathbf{y}'_t(\hat{\mathbb{B}})(\ \mathbb{B} - \hat{\mathbb{B}}\ ) + \mathbf{e}'_t(\mathbb{B})$$

$$\mathbb{B}' = [\Phi, \Theta] \quad , \quad \mathbf{y}'_t(\mathbb{B}) = [\mathbf{z}'_{t-1}, -\mathbf{e}'_{t-1}(\mathbb{B})]$$

moreover having

$$\hat{\mathbf{e}}'_t(k) = \mathbf{z}'_t - \hat{\mathbf{y}}'_t(k)\ \hat{\mathbb{B}}(k)$$

the I-PLR estimator of the $ARMA_m(1,1)$ reduces to

*(I-PLR)* $$\hat{\mathbb{B}}(k+1) = [\Sigma_{t=1}^{n}\ \hat{\mathbf{y}}_t(k)\hat{\mathbf{y}}'_t(k)]^{-1}\ \Sigma_{t=1}^{n}\ \hat{\mathbf{y}}_t(k)\ \mathbf{z}'_t$$

Generalizing the analysis of Ljung-Söderström(1983), we may state that this algorithm is consistent if $\Theta(z)^{-1}$ is passive :

$$|\ \Theta(z)^{-1} - \tfrac{1}{2}\mathbb{I}\ | > 0 \quad , \ |z| \leq 1$$

Notice that since Det $\Theta(B)$ has degree $(m^2q)$, the condition is not easy to sati̲sfy for $(m,q) > 2$ . A simple necessary condition for the above is however provided by the passivity of the monic polynomials $\theta_{ii}(B)$ on the principal diagonal.

Finally, in presence of simultaneous correlation, an efficient system-estima̲tor, which also yields a joint estimation of all the regression coefficients $(\Phi, \Theta, \Sigma)$ is given by the seemingly unrelated structure

$$\beta = \text{Vec}\,[\mathbb{B}'] \quad , \quad \mathbb{Y}' = [\mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_n]$$

$$\hat{\beta}_{k+1} = \{[\mathbb{I}_m \otimes \hat{\mathbb{Y}}_k]'\,[\hat{\Sigma}_k \otimes \mathbb{I}_n]^{-1}\,[\mathbb{I}_m \otimes \hat{\mathbb{Y}}_k]\}^{-1}\,[\mathbb{I}_m \otimes \hat{\mathbb{Y}}_k]'\,[\hat{\Sigma}_k \otimes \mathbb{I}_n]^{-1}\ \mathbf{z}$$

This structure is justified by the fact that each row-equation of the $ARMA_m$ has the same set of pseudolinear regressors $\mathbf{y}_t(\mathbb{B})$ .

## 4. EMPIRICAL COMPARISONS Of TFS - $ARMA_m$

The economic problem considered for the empirical comparisons is the analysis of the foreign sources of the price inflation in Italy. We define 5 variables :

$ = exchange rate Lira/Dollar ,
PI = ISTAT index of wholesale prices ,
PX = " " " export " ,   t = monthly data 1973.1 - 1985.12
PM = " " " import " ,
B = balance of foreign trade ,

### *(4.1) Serial Correlation Analysis*

All the processes graphically have evidenced components of trends. The analysis of the variances and of the correlograms on differenced series shows that stationarity may be reached with a differentiation of order one for all the variables.

The plots of the sample correlation functions are reported in Figure 1 and 2. Here, we may note the high simultaneous correlation of the prices due to the fact PX and PM are the prices of the exported and imported goods, so that PI,PX, PM are synonymous. The processes $(1-B)PI_t$ and $(1-B)B_t$, again exhibit a considerable autocorrelation (of AR(1) and MA(1) type) , while the other series are practically white noises. To identify the functions $v_{ij}(B)$, an analysis on prewhitened series is suggested ; the corresponding univariate filters are :

$$\underset{(.065)}{(1+.621\,B)}(1-B)\ PI_t = pi_t$$

$$(1-B)\ B_t = \underset{(.053)}{(1-.763\,B)}\ b_t$$

The cross correlograms computed on the prewhitened series are reported in Figure 3 . Here, we can get an empirical evidence of what was said in the introduction about identification and polynomial orthogonality : since $(1-B)B_t \sim MA(1)$ and b>0, the CCRF(B,PM) does not change after prewhitening ; the same is not true for the CCRF(PI,PM) because $(1-B)PI_t \sim AR(1)$ and b=0 .

### *(4.2) Identification, Estimation of $ARMA_5$*

The identification strategies of the $ARMA_m$, see Tiao-Box(1981), Jenkins-Alavi(1981), are consequent on the *genesis* of the structure of that model .

As we said, the multivariate spectral factorization theorem does not ensure

FIG. 1 - Sample $CCRF_s$ (diffenced series)

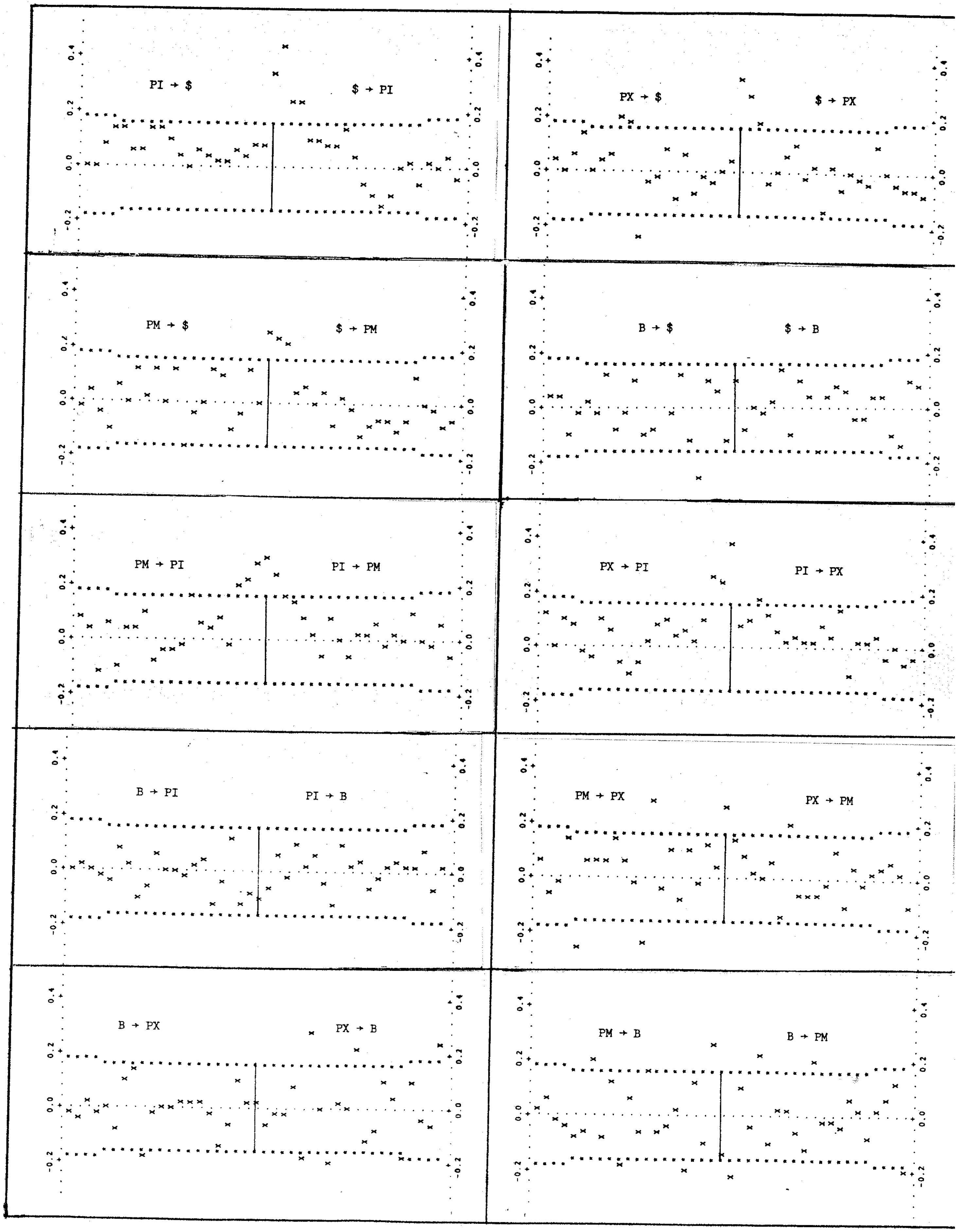

FIG. 1' - Sample Cross Correlation Functions (Differenced Series)

| LAG | $←PI | $←PX | $←PM | $←B | PI←PX | PI←PM | PI←B | PX←PM | PX←B | PM←B |
|---|---|---|---|---|---|---|---|---|---|---|
| -20 | 0.00 | 0.03 | -0.01 | 0.04 | 0.12 | 0.09 | 0.00 | 0.07 | -0.03 | 0.02 |
| -19 | -0.01 | 0.00 | 0.04 | 0.04 | -0.01 | 0.04 | 0.02 | -0.07 | -0.05 | 0.06 |
| -18 | 0.08 | 0.06 | -0.05 | -0.10 | 0.11 | -0.12 | 0.00 | -0.01 | 0.02 | -0.01 |
| -17 | 0.13 | 0.14 | -0.09 | -0.02 | 0.08 | 0.05 | -0.02 | 0.15 | -0.03 | -0.03 |
| -16 | 0.14 | 0.01 | 0.05 | 0.03 | -0.02 | -0.10 | -0.03 | -0.25 | -0.01 | -0.08 |
| -15 | 0.05 | 0.04 | -0.01 | -0.01 | -0.04 | 0.05 | 0.08 | 0.06 | -0.09 | -0.07 |
| -14 | 0.05 | 0.05 | 0.12 | 0.12 | 0.09 | 0.05 | 0.02 | 0.06 | 0.10 | 0.21 |
| -13 | 0.14 | 0.20 | 0.02 | -0.09 | 0.05 | 0.10 | -0.11 | 0.06 | 0.15 | -0.09 |
| -12 | 0.14 | 0.18 | 0.12 | -0.02 | -0.06 | -0.09 | -0.05 | 0.13 | -0.18 | 0.11 |
| -11 | 0.10 | -0.24 | 0.00 | 0.10 | -0.11 | -0.04 | 0.06 | 0.06 | -0.02 | -0.18 |
| -10 | 0.04 | -0.03 | 0.12 | -0.10 | -0.06 | -0.05 | 0.01 | -0.03 | 0.01 | 0.05 |
| -9 | -0.01 | -0.02 | -0.16 | -0.07 | 0.01 | -0.02 | 0.01 | -0.24 | 0.00 | -0.05 |
| -8 | 0.06 | 0.07 | -0.04 | 0.17 | 0.09 | 0.16 | -0.01 | 0.29 | 0.02 | 0.15 |
| -7 | 0.04 | -0.10 | 0.00 | -0.02 | 0.09 | 0.05 | 0.02 | -0.04 | 0.03 | -0.06 |
| -6 | 0.01 | 0.06 | 0.11 | 0.13 | 0.04 | 0.03 | 0.04 | 0.09 | 0.02 | -0.05 |
| -5 | 0.02 | -0.07 | 0.10 | -0.11 | 0.07 | 0.07 | -0.12 | -0.08 | -0.02 | 0.01 |
| -4 | 0.06 | -0.01 | -0.09 | -0.26 | 0.02 | -0.01 | -0.04 | 0.10 | -0.14 | -0.20 |
| -3 | 0.05 | -0.05 | -0.04 | 0.10 | 0.10 | 0.20 | 0.11 | -0.01 | -0.05 | 0.13 |
| -2 | 0.08 | 0.01 | 0.12 | 0.07 | 0.25 | 0.22 | -0.12 | 0.11 | 0.11 | -0.10 |
| -1 | 0.07 | 0.03 | 0.00 | -0.11 | 0.25 | 0.27 | -0.08 | 0.04 | 0.02 | 0.26 |
| 0 | 0.34 | 0.35 | 0.26 | 0.11 | 0.38 | 0.31 | -0.09 | 0.26 | 0.02 | -0.14 |
| 1 | 0.45 | 0.27 | 0.25 | -0.08 | 0.08 | 0.24 | -0.05 | 0.15 | -0.05 | -0.22 |
| 2 | 0.24 | 0.18 | 0.22 | 0.01 | 0.10 | 0.16 | 0.05 | 0.08 | -0.02 | 0.10 |
| 3 | 0.23 | -0.03 | 0.04 | -0.02 | 0.18 | 0.13 | -0.02 | 0.01 | -0.02 | -0.05 |
| 4 | 0.09 | 0.00 | 0.06 | 0.02 | 0.11 | 0.08 | 0.10 | -0.01 | 0.07 | 0.23 |
| 5 | 0.10 | 0.05 | 0.00 | 0.13 | 0.06 | 0.03 | 0.03 | 0.07 | -0.19 | -0.10 |
| 6 | 0.08 | 0.10 | 0.04 | -0.10 | 0.02 | -0.06 | 0.05 | -0.15 | 0.28 | 0.01 |
| 7 | 0.07 | -0.02 | -0.07 | 0.10 | 0.05 | 0.08 | -0.03 | 0.21 | 0.00 | 0.01 |
| 8 | 0.14 | 0.03 | 0.03 | 0.04 | 0.02 | 0.01 | -0.13 | -0.06 | -0.19 | -0.06 |
| 9 | 0.04 | -0.13 | -0.01 | -0.15 | 0.01 | -0.07 | 0.09 | -0.06 | 0.02 | -0.11 |
| 10 | -0.07 | 0.01 | -0.11 | 0.04 | 0.09 | 0.03 | 0.01 | -0.06 | -0.01 | 0.20 |
| 11 | -0.10 | -0.07 | -0.08 | 0.09 | 0.04 | 0.01 | 0.03 | -0.02 | 0.22 | -0.02 |
| 12 | -0.14 | -0.01 | -0.06 | 0.06 | 0.14 | 0.07 | -0.07 | 0.10 | -0.11 | -0.03 |
| 13 | -0.10 | -0.02 | -0.05 | -0.04 | -0.10 | -0.02 | -0.02 | -0.09 | -0.08 | -0.04 |
| 14 | 0.00 | -0.05 | -0.10 | -0.04 | 0.02 | 0.03 | 0.02 | 0.04 | 0.09 | 0.06 |
| 15 | 0.02 | 0.10 | -0.05 | 0.05 | 0.03 | -0.01 | 0.04 | -0.02 | 0.04 | 0.01 |
| 16 | -0.06 | 0.01 | 0.10 | 0.03 | 0.03 | 0.09 | 0.01 | 0.03 | -0.18 | -0.08 |
| 17 | 0.02 | -0.03 | 0.01 | -0.10 | -0.04 | -0.01 | 0.02 | 0.04 | 0.11 | 0.01 |
| 18 | 0.00 | -0.06 | -0.02 | -0.14 | 0.01 | -0.01 | 0.08 | 0.07 | -0.04 | 0.06 |
| 19 | 0.04 | -0.06 | -0.07 | 0.11 | -0.06 | 0.06 | -0.06 | 0.01 | -0.06 | 0.13 |
| 20 | -0.04 | -0.08 | -0.06 | 0.08 | -0.04 | -0.06 | 0.02 | -0.09 | 0.24 | -0.19 |
| | $→PI | $→PX | $→PM | $→B | PI→PX | PI→PM | PI→B | PX→PM | PX→B | PM→B |

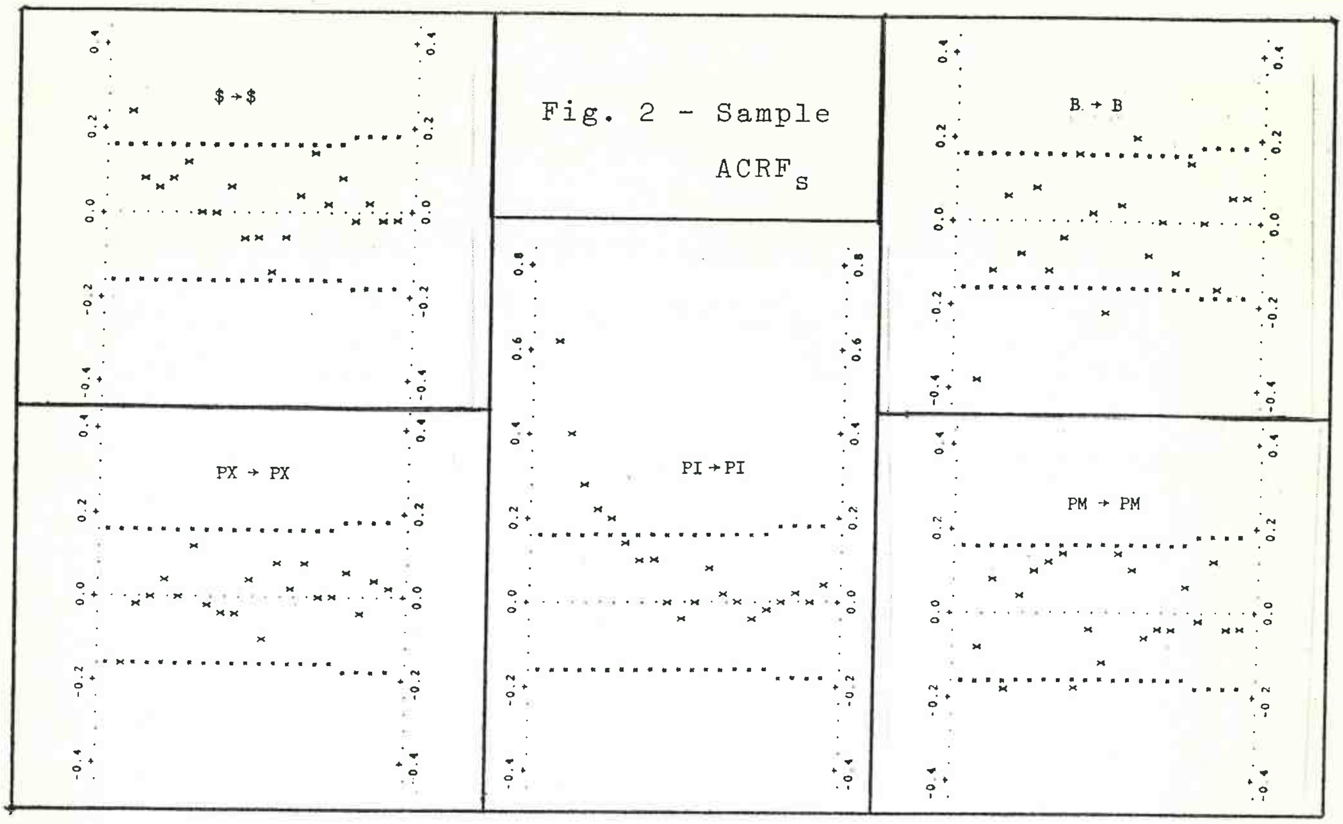


Fig. 2 - Sample $ACRF_s$

FIG. 3 - Sample $CCRF_s$ (prewhitened series)

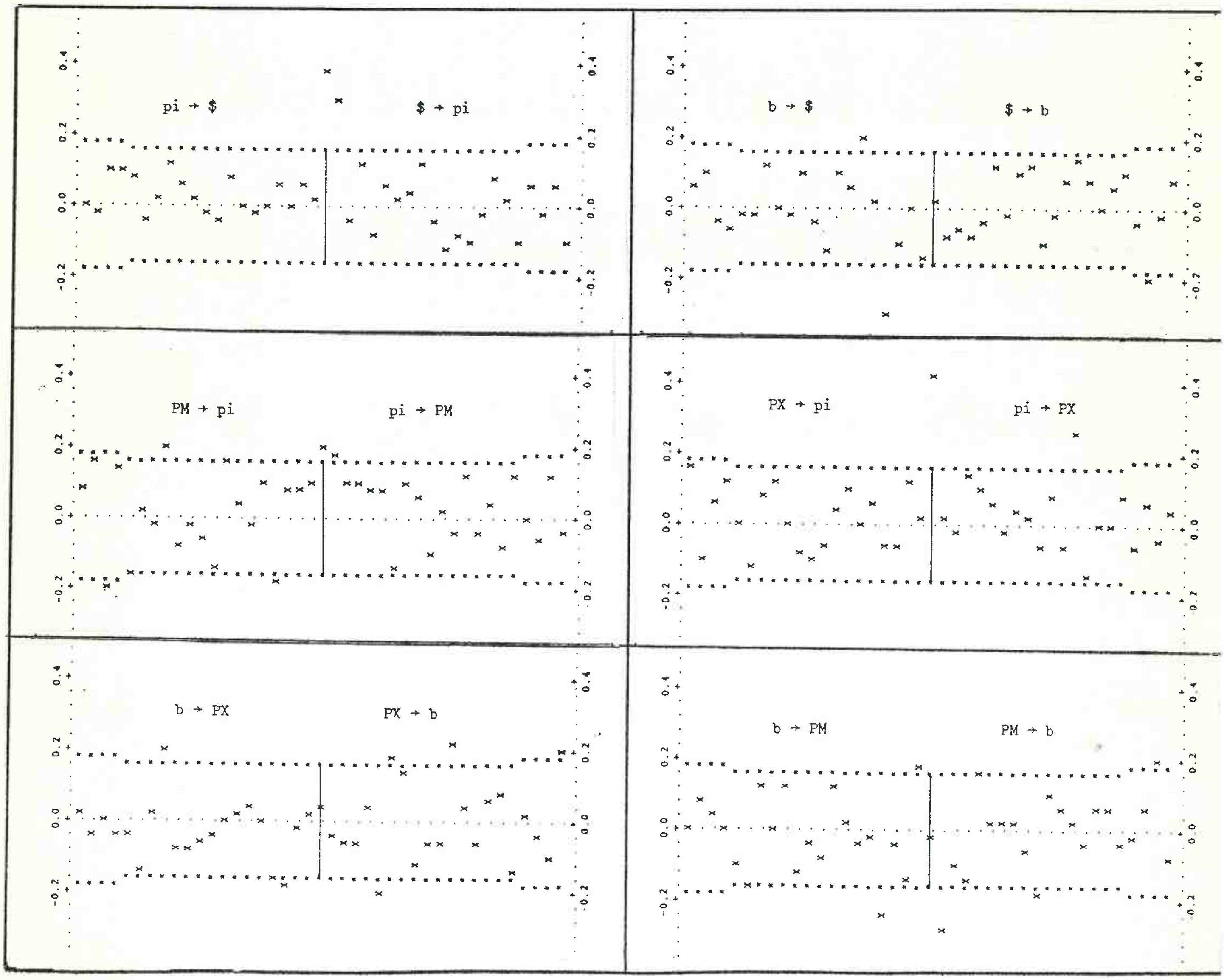

FIG. 2' - Sample Autocorrelation Functions (Differnced Series)

| LAG | $ | PI | PX | PM | B |
|---|---|---|---|---|---|
| 0 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 1 | 0.24 | 0.62 | -0.16 | -0.09 | -0.38 |
| 2 | 0.07 | 0.41 | -0.02 | 0.08 | -0.11 |
| 3 | 0.05 | 0.28 | 0.01 | -0.18 | 0.07 |
| 4 | 0.08 | 0.23 | 0.04 | 0.04 | -0.09 |
| 5 | 0.12 | 0.20 | 0.00 | 0.09 | 0.08 |
| 6 | 0.00 | 0.14 | 0.11 | 0.12 | -0.11 |
| 7 | 0.01 | 0.11 | -0.01 | 0.14 | -0.04 |
| 8 | 0.06 | 0.09 | -0.03 | -0.18 | 0.16 |
| 9 | -0.05 | 0.00 | -0.04 | -0.04 | 0.02 |
| 10 | -0.06 | -0.03 | 0.04 | -0.12 | -0.21 |
| 11 | -0.13 | 0.00 | -0.10 | 0.15 | 0.03 |
| 12 | -0.06 | 0.08 | 0.09 | 0.10 | 0.21 |
| 13 | 0.05 | 0.02 | 0.03 | -0.06 | -0.07 |
| 14 | 0.14 | 0.01 | 0.07 | -0.05 | 0.00 |
| 15 | 0.03 | -0.04 | -0.01 | -0.03 | -0.12 |
| 16 | 0.08 | -0.01 | 0.00 | 0.06 | 0.14 |
| 17 | -0.03 | 0.00 | 0.05 | -0.02 | -0.01 |
| 18 | 0.02 | 0.02 | -0.05 | 0.12 | -0.15 |
| 19 | -0.03 | 0.00 | 0.04 | -0.05 | 0.05 |
| 20 | -0.03 | 0.04 | 0.01 | -0.04 | 0.06 |

FIG. 3' - Sample Cross Correlation Functions (Prewhitened Series)

| LAG | $←pi | $←b | pi←px | pi←pm | pi←b | px←b | pm←b |
|---|---|---|---|---|---|---|---|
| -20 | 0.01 | 0.06 | 0.16 | 0.08 | -0.01 | 0.01 | 0.01 |
| -19 | -0.01 | 0.10 | -0.10 | 0.15 | 0.01 | -0.04 | 0.07 |
| -18 | 0.11 | -0.04 | 0.07 | -0.20 | 0.02 | -0.01 | 0.04 |
| -17 | 0.11 | -0.06 | 0.12 | 0.14 | 0.03 | -0.04 | -0.01 |
| -16 | 0.07 | -0.01 | 0.01 | -0.16 | -0.10 | -0.04 | -0.10 |
| -15 | -0.05 | -0.03 | -0.12 | 0.02 | 0.02 | -0.13 | -0.16 |
| -14 | 0.03 | 0.12 | 0.08 | -0.02 | 0.17 | 0.02 | 0.12 |
| -13 | 0.13 | -0.01 | 0.12 | 0.20 | 0.00 | 0.19 | -0.01 |
| -12 | 0.07 | -0.03 | -0.01 | -0.08 | -0.13 | -0.07 | 0.12 |
| -11 | 0.02 | 0.09 | -0.08 | -0.02 | -0.01 | -0.08 | -0.12 |
| -10 | -0.02 | -0.04 | -0.10 | -0.05 | -0.01 | -0.05 | -0.03 |
| -9 | -0.04 | -0.12 | -0.05 | -0.15 | 0.02 | -0.04 | -0.09 |
| -8 | 0.09 | 0.11 | 0.04 | 0.16 | -0.02 | -0.01 | 0.11 |
| -7 | -0.01 | 0.06 | 0.09 | 0.05 | -0.02 | 0.02 | 0.02 |
| -6 | -0.01 | 0.20 | -0.01 | -0.01 | 0.13 | 0.04 | -0.04 |
| -5 | 0.01 | 0.01 | 0.07 | 0.10 | -0.04 | 0.00 | -0.02 |
| -4 | 0.07 | -0.30 | -0.06 | -0.17 | -0.19 | -0.16 | -0.25 |
| -3 | 0.01 | -0.11 | -0.06 | 0.08 | 0.13 | -0.19 | -0.04 |
| -2 | 0.07 | 0.00 | 0.13 | 0.07 | 0.02 | -0.02 | -0.15 |
| -1 | 0.02 | -0.14 | 0.02 | 0.11 | -0.02 | 0.02 | 0.19 |
| 0 | 0.37 | 0.02 | 0.41 | 0.20 | -0.11 | 0.04 | -0.03 |
| 1 | 0.31 | -0.08 | 0.02 | 0.18 | -0.21 | -0.04 | -0.28 |
| 2 | -0.04 | -0.06 | -0.02 | 0.10 | -0.07 | -0.05 | -0.10 |
| 3 | 0.12 | -0.08 | 0.14 | 0.10 | -0.18 | -0.07 | -0.13 |
| 4 | -0.07 | -0.04 | 0.09 | 0.08 | -0.01 | 0.03 | 0.16 |
| 5 | 0.06 | 0.12 | 0.07 | 0.09 | -0.01 | -0.19 | 0.01 |
| 6 | 0.02 | -0.02 | -0.02 | -0.14 | 0.09 | 0.19 | 0.02 |
| 7 | 0.04 | 0.10 | 0.05 | 0.09 | 0.14 | 0.14 | 0.03 |
| 8 | 0.11 | 0.12 | 0.01 | 0.06 | -0.17 | -0.12 | -0.05 |
| 9 | -0.05 | -0.09 | -0.06 | -0.11 | -0.01 | -0.07 | -0.18 |
| 10 | -0.12 | -0.03 | 0.09 | 0.03 | -0.01 | -0.06 | 0.10 |
| 11 | -0.09 | 0.09 | -0.06 | -0.04 | 0.10 | 0.21 | 0.06 |
| 12 | -0.10 | 0.15 | 0.26 | 0.11 | 0.00 | 0.04 | 0.02 |
| 13 | -0.02 | 0.07 | -0.14 | -0.05 | -0.05 | -0.06 | -0.03 |
| 14 | 0.08 | 0.01 | 0.00 | 0.04 | -0.06 | 0.06 | 0.05 |
| 15 | 0.02 | 0.07 | 0.01 | -0.09 | 0.01 | 0.08 | 0.06 |
| 16 | -0.10 | 0.10 | 0.08 | 0.12 | 0.01 | -0.14 | -0.05 |
| 17 | 0.07 | -0.03 | -0.06 | 0.00 | -0.04 | 0.02 | -0.02 |
| 18 | -0.02 | -0.19 | 0.06 | -0.06 | 0.15 | -0.03 | 0.06 |
| 19 | 0.05 | -0.02 | -0.05 | 0.12 | 0.01 | -0.09 | 0.20 |
| 20 | -0.09 | 0.08 | 0.03 | -0.04 | -0.02 | 0.21 | -0.08 |
| | $→pi | $→b | pi→px | pi→pm | pi→b | px→b | pm→b |

the existence of a *general* $ARMA_m$ representation, but only of a particular canonical form with one of the two matrices $\Phi(B)$, $\Theta(B)$ taken diagonal. Indeed, given

$$\mathbb{z}_t = \Psi(B)\, \mathbb{e}_t \quad , \qquad \Psi(z) = \sqrt{\Gamma}(z)$$

Hannan(1979)p.85, has only suggested as further factorization for $\Psi(z)=\{\psi_{ij}(z)\}$ the least common denominator (LCD)

$$\phi(z) = \mathrm{LCD}\{\psi_{ij}(z)\} \quad \Rightarrow \quad \begin{aligned} \dot{\Phi}(z) &= \mathbb{I}_m \cdot \phi(z) \\ \tilde{\Theta}(z) &= \phi(z)\cdot\Psi(z) \end{aligned}$$

A less rough technic may consider a linearization by row , but the substance of the problem remains unchanged .

The genesis of the $ARMA_m$ structure follows, then, a *superficial* generalization of the univariate ARMA model, and in this extension it does not consider the different and autonomous *nature* of the cross-correlation with respect the auto-correlation (in particular $\rho_{ij}(0)\neq 1$, $\rho_{ij}(k)\neq\rho_{ij}(-k)$, b>1, and so on).

It is true that auto-relationships are more powerful and significant than cross ones ; however, in the $ARMA_m$ context, the cross-correlation is treated as a trivial projection of the auto-correlation. Following these considerations a coherent strategy of identification would seem to be

$$p = \max(p_i) \quad , \quad q = \max(q_i) \quad , \qquad i = 1,2 \ldots m$$

where $(p_i, q_i)$ are the orders of the univariate ARMA models of the series $\{z_{i_t}\}$.

In our data we identify an $ARMA_5(1,1)$ model. The practical implementation of the pseudolinear estimation has followed these steps :

0) Estimate an $AR_5(3)$ : $\Phi_k(0) \quad k = 1,2,3$
   Generate $\hat{\mathbb{e}}_t(0) = \mathbb{z}_t - \Sigma_{k=1}^{3} \hat{\Phi}_k(0)\mathbb{z}_{t-k}$

1) Estimate the $ARMA_5(1,1)$ : $\mathbb{z}_t = \Phi(1)\mathbb{z}_{t-1} + \Theta(1)\hat{\mathbb{e}}_{t-1}(0) + \tilde{\mathbb{e}}_t(1)$
   Generate $\hat{\mathbb{e}}_t(1) = \mathbb{z}_t - \hat{\Phi}(1)\mathbb{z}_{t-1} - \hat{\Theta}(1)\hat{\mathbb{e}}_{t-1}(1)$

2) Estimate the $ARMA_5(1,1)$ : $\mathbb{z}_t = \Phi(2)\mathbb{z}_{t-1} + \Theta(2)\hat{\mathbb{e}}_{t-1}(1) + \tilde{\mathbb{e}}_t(2)$
   And so on ...

In a first estimation the algorithm has not converged owing to the high simultaneous correlation, the great number of parameters to be estimated (25+25+15=75) and the non-significance of many $\phi_{ij}$, $\theta_{ij}$. The last two situations have probably caused the non-passivity of Det$\Theta$(B), although the specific fact responsible of the divergence was $\hat{\theta}_{55}(k) \rightarrow 1$.

A simplification of the model, eliminating all the non-significant coefficients (starting from the third iteration) has improved the situation. In 15 iterations, with constant stepsize ($\frac{1}{2}$), convergence was achieved ; results are in Table 1 .

$$\begin{bmatrix} \$_t \\ PI_t \\ PX_t \\ PM_t \\ B_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & \phi_{13} & 0 & 0 \\ 0 & \phi_{22} & 0 & 0 & 0 \\ 0 & 0 & \phi_{33} & 0 & 0 \\ \phi_{41} & 0 & 0 & 0 & \phi_{45} \\ 0 & \phi_{52} & 0 & \phi_{54} & 0 \end{bmatrix} \begin{bmatrix} \$_{t-1} \\ PI_{t-1} \\ PX_{t-1} \\ PM_{t-1} \\ B_{t-1} \end{bmatrix} + \begin{bmatrix} \theta_{11} & 0 & \theta_{13} & 0 & 0 \\ 0 & 0 & 0 & \theta_{24} & 0 \\ \theta_{31} & 0 & \theta_{33} & 0 & 0 \\ 0 & 0 & 0 & \theta_{44} & 0 \\ 0 & \theta_{52} & 0 & 0 & \theta_{55} \end{bmatrix} \begin{bmatrix} \$_{t-1} \\ pi_{t-1} \\ px_{t-1} \\ pm_{t-1} \\ b_{t-1} \end{bmatrix} + \begin{bmatrix} \$_t \\ pi_t \\ px_t \\ pm_t \\ b_t \end{bmatrix}$$

Table 1 - $ARMA_m$ Estimates ($P=\phi$, $Q=\theta$)

| PARAMETER | ESTIMATE | STANDARD ERROR | T-STATISTIC |
|---|---|---|---|
| P113 | 1.283722 | 0.6274935 | 2.045793 |
| Q111 | 0.1296638 | 0.7828205E-01 | 1.656367 |
| Q113 | -1.695211 | 0.6899980 | -2.456835 |
| P122 | 0.5042209 | 0.5826713E-01 | 8.653608 |
| Q124 | 0.1020208E-01 | 0.4063113E-02 | 2.510903 |
| P133 | 0.3173806 | 0.1725847 | 1.838984 |
| Q131 | 0.7307190E-01 | 0.2152783E-01 | 3.394300 |
| Q133 | -0.6997944 | 0.1898797 | -3.685463 |
| P141 | 0.1247098 | 0.4832006E-01 | 2.580912 |
| P145 | 5.900188 | 1.651278 | 3.573103 |
| Q144 | -0.1600814 | 0.7938240E-01 | -2.016586 |
| P152 | 0.1302421 | 0.7757674E-01 | 1.678881 |
| P154 | -0.1083171E-01 | 0.2978710E-02 | -3.636375 |
| Q152 | -0.2086790 | 0.9814490E-01 | -2.126234 |
| Q155 | -0.6660523 | 0.8030530E-01 | -8.294002 |

RESIDUAL COVARIANCE MATRIX

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 832.22998 | 10.47000 | 70.06000 | 136.87000 | 0.98000 |
| 2 | 10.47000 | 0.92000 | 3.26000 | 3.79000 | -0.05000 |
| 3 | 70.06000 | 3.26000 | 65.81000 | 37.96000 | 0.46000 |
| 4 | 136.87000 | 3.79000 | 37.96000 | 305.95001 | -0.74000 |
| 5 | 0.98000 | -0.05000 | 0.46000 | -0.74000 | 0.44000 |

### *(4.3) Identification, Estimation of $TFS_5$*

The multivariate extension given by the TFS tries to respect the different nature of the cross-relationships for which the Box-Jenkins methodology has provided autonomous apparatus of modeling and identification. We have shown that under adequate stationarity certain conditions of polynomial orthogonality enable simplified MA-representation and spectral factorization for the TFS. With these we have defined a *disaggregate* strategy of identification which directly extends the Box-Jenkins schemes.

Reasoning on Figure 1,2,3 we have identified the model

$$\begin{bmatrix} 1 & 0 & \left(\frac{-\omega_0+\omega_1 B+\omega_2 B^2}{1+\delta_1 B}\right)B^{11} & 0 & \left(\frac{-\omega_0}{1-\delta_1 B^4}\right)B^4 \\ \left(\frac{\omega_0}{1+\delta_1 B^2}\right)B & 1 & 0 & \left(\frac{-\omega_0}{1-\delta_1 B^4}\right)B^4 & 0 \\ \left(\frac{\omega_0}{1+\delta_1 B}\right)B & \omega_0 B^3 & 1 & \left(\frac{\omega_0}{1-\delta_1 B^8}\right)B^8 & 0 \\ \left(\frac{\omega_0}{1+\delta_1 B}\right)B & \omega_0 B & \omega_0 B^7 & 1 & \left(\frac{\omega_0}{1-\delta_1 B^3}\right)B \\ 0 & 0 & \left(\frac{\omega_0}{1-\delta_1 B^2}\right)B^6 & \left(\frac{-\omega_0}{1-\delta_1 B^3}\right)B & 1 \end{bmatrix} \begin{bmatrix} \$_t \\ PI_t \\ PX_t \\ PM_t \\ B_t \end{bmatrix} = \begin{bmatrix} (1+\theta_1 B)\ \$_t \\ \left(\frac{1}{1+\phi_1 B}\right) pi_t \\ (1-\theta_1 B)\ px_t \\ (1-\theta_1 B^3) pm_t \\ (1-\theta_1 B)\ b_t \end{bmatrix}$$

many $v_{ij}(B)$ are at the limit of the Box-Jenkins identification but this forcing was necessary to test the performance of the pseudolinear estimators.

In effect, for the function $v_{24}(B)$ an *ad-hoc* search analysis (fixing all the other coefficients) was required to find the narrow band of convergence. Without stepsize ($\frac{1}{2}$) other forced impulse response functions, as $v_{34}(B)$, diverge in an oscillatory manner. After 7 iterations, using method (3.3) for the initial values, convergence was achieved ; results are in Table 2 .

The validity of the disaggregate identification is pointed out by the statistical significance of the estimates and by the fact that their signs coincide with that expected from the analysis of the correlograms .

Empirical check of the orthogonal polynomial approximation is obtained by estimating the TFS in the simplified AR-form $[\dot{\mathbb{I}}(B) - \mathbb{V}(B)]\, \mathbb{z}_t = \mathbb{a}_t$, by means of the second pseudolinear algorithm . After 10 iterations, using as initial va-

Table 2 - TFS Estimates (D=δ, O=ω, T=θ, P=ϕ)

| PARAMETER | ESTIMATE | STANDARD ERROR | T-STATISTIC |
|---|---|---|---|
| D13 | 0.3872898 | 0.2113881 | 1.832127 |
| O130 | -0.8257078 | 0.2298072 | -3.593046 |
| O131 | 0.8619491 | 0.2938870 | 2.932927 |
| O132 | 0.9357828 | 0.2493754 | 3.752506 |
| D15 | -0.6580830 | 0.3069839 | -2.143705 |
| O15 | -5.872073 | 2.556939 | -2.296524 |
| TH1 | 0.2315633 | 0.8211036E-01 | 2.820148 |
| D21 | 0.6248492 | 0.1585592 | 3.940794 |
| O21 | 0.1113244E-01 | 0.2502362E-02 | 4.448773 |
| D24 | -0.8481432 | 0.4799062 | -1.767310 |
| O24 | -0.5106808E-02 | 0.3496731E-02 | -1.460452 |
| PH2 | 0.5409124 | 0.6317930E-01 | 8.561545 |
| D31 | 0.1433220 | 0.2041528 | 0.7020332 |
| O31 | 0.9892479E-01 | 0.2281421E-01 | 4.336105 |
| O32 | 0.8736253 | 0.4709073 | 1.855196 |
| D34 | -0.4984056 | 0.2230868 | -2.234133 |
| O34 | 0.1138837 | 0.3161667E-01 | 3.602014 |
| TH3 | -0.3410596 | 0.7921210E-01 | -4.305650 |
| D41 | 0.5560457 | 0.4633728 | 1.199996 |
| O41 | 0.8315278E-01 | 0.4917492E-01 | 1.690959 |
| O42 | 2.811315 | 1.283130 | 2.190981 |
| O43 | 0.3408410 | 0.1557554 | 2.188309 |
| D45 | -0.4684858 | 0.2359980 | -1.985126 |
| O45 | 6.525995 | 1.677477 | 3.890363 |
| TH4 | -0.2157409 | 0.7851432E-01 | -2.747790 |
| D53 | -0.5902944 | 0.2683395 | -2.199805 |
| O53 | 0.1917398E-01 | 0.6018044E-02 | 3.186082 |
| D54 | -0.5010504 | 0.2476550 | -2.023179 |
| O54 | -0.8816265E-02 | 0.2885914E-02 | -3.054930 |
| TH5 | -0.5997447 | 0.8332490E-01 | -7.197665 |

RESIDUAL COVARIANCE MATRIX

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 631.47999 | 6.41294 | 40.60615 | 112.27674 | 2.56567 |
| 2 | 6.41294 | 0.79427 | 2.67697 | 3.65158 | -0.06033 |
| 3 | 40.60615 | 2.67697 | 55.32032 | 36.62695 | 0.26424 |
| 4 | 112.27674 | 3.65158 | 36.62695 | 265.57731 | -0.90807 |
| 5 | 2.56567 | -0.06033 | 0.26424 | -0.90807 | 0.39353 |

Table 3 - $RAR_m$ Estimates

| PARAMETER | ESTIMATE | STANDARD ERROR | T-STATISTIC |
|---|---|---|---|
| D13 | 0.1085634 | 0.2891577 | 0.3754471 |
| O130 | -0.8310237 | 0.2301243 | -3.611195 |
| O131 | 0.8521420 | 0.3368120 | 2.530023 |
| O132 | 0.7989285 | 0.2925673 | 2.730751 |
| D15 | -0.7085768 | 0.2969198 | -2.386425 |
| O15 | -5.822430 | 2.553287 | -2.280367 |
| TH1 | 0.2536032 | 0.7880366E-01 | 3.218165 |
| D21 | 0.5809622 | 0.1903071 | 3.052762 |
| O21 | 0.1016894E-01 | 0.2579238E-02 | 3.942614 |
| D24 | -0.7293874 | 0.3337230 | -2.185607 |
| O24 | -0.8782266E-02 | 0.3586899E-02 | -2.448429 |
| PH2 | 0.4771779 | 0.6032870E-01 | 7.909633 |
| D31 | 0.4815813 | 0.1782003 | 2.702472 |
| O31 | 0.1001763 | 0.2224433E-01 | 4.503452 |
| O32 | 0.9934272 | 0.4733815 | 2.098576 |
| D34 | -0.4170122 | 0.2110785 | -1.975626 |
| O34 | 0.1242945 | 0.3139830E-01 | 3.958639 |
| TH3 | -0.3663954 | 0.7206808E-01 | -5.084019 |
| D41 | 0.5796921 | 0.4138023 | 1.400891 |
| O41 | 0.8847462E-01 | 0.4865752E-01 | 1.818313 |
| O42 | 3.061703 | 1.280497 | 2.391027 |
| O43 | 0.3849985 | 0.1534577 | 2.508825 |
| D45 | -0.3341113 | 0.2532441 | -1.319325 |
| O45 | 6.559011 | 1.656162 | 3.960368 |
| TH4 | -0.2404554 | 0.7309620E-01 | -3.289575 |
| D53 | -0.4656491 | 0.3064413 | -1.519538 |
| O53 | 0.1917188E-01 | 0.6297327E-02 | 3.044447 |
| D54 | -0.4455631 | 0.2784486 | -1.600163 |
| O54 | -0.8593774E-02 | 0.3008144E-02 | -2.856836 |
| TH5 | -0.4750252 | 0.7787804E-01 | -6.099605 |

S

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 627.35180 | 5.89659 | 39.03565 | 111.61042 | 2.78974 |
| 2 | 5.89659 | 0.80308 | 2.73084 | 3.55253 | -0.05286 |
| 3 | 39.03565 | 2.73084 | 53.55003 | 35.00476 | 0.15817 |
| 4 | 111.61042 | 3.55253 | 35.00476 | 258.90521 | -0.81612 |
| 5 | 2.78974 | -0.05286 | 0.15817 | -0.81612 | 0.42233 |

lues the estimates of Table 2 and, again, with an *ad-hoc* search for $v_{24}(B)$, we obtained the results of Table 3 .

The defferences between the estimates of Table 2, 3 are not important; the fact that $\hat{\delta}_{31}(B) \to 1$ in Tab.2 is compensated by $\hat{\delta}_{13}(B) \to 1$ in Tab.3 . The redundancy of the polynomials $\delta_{ij}(B), \theta_i(B)$ probably lies at the origin of the two situations, a simplification of the model is in order.

### *(4.4) Reunification of $ARMA_m$ and TFS*

In the previous section many rational functions v(B) were at the limit of identification. A linear modeling like $v_{ij}(B) = \omega_{ij}(B)$ is more realistic and flexible, and also improves the speed of convergence of the estimation algorithms.

The model that follows looks like a closed-loop system of simultaneous ARMAX equations and provides a substantial reunification of the $ARMA_m$,TFS structures

$$\phi_i(B)\, z_{i_t} + (\Sigma_{j \neq i}^{m}\ \omega_{ij}(B)\, z_{j_{t-b}}) = \theta_i(B)\, a_{i_t}$$

*($ARMAX_m$)*
$$[\dot{\Phi}(B) - \mathbb{\Omega}(B)]\ \mathbb{z}_t = \dot{\Theta}(B)\ \mathbb{a}_t$$

The $\omega_{ij}(k)$ coefficients are identified in the same position as the significant cross correlations coefficients. Estimation results are given in Table 4 .

$$\begin{bmatrix} 1 & 0 & (-\omega_1B^{11}+\omega_2B^{12}+\omega_3B^{13}) & 0 & (-\omega_1B^4+\omega_2B^8) \\ (\omega_1B+\omega_2B^3) & (1+\phi_1B) & 0 & (-\omega_1B^4+\omega_2B^8) & 0 \\ (\omega_1B+\omega_2B^2) & (\omega_1B^3+\omega_2B^{12}) & 1 & (\omega_1B^8-\omega_2B^9-\omega_3B^{16}) & 0 \\ (\omega_1B+\omega_2B^2) & (\omega_1B-\omega_2B^6) & \omega_1B^7 & 1 & (\omega_1B-\omega_2B^4) \\ 0 & 0 & (-\omega_1B^5+\omega_2B^6-\omega_3B^8+\omega_4B^{11}) & (-\omega_1B+\omega_2B^4) & 1 \end{bmatrix} \begin{bmatrix} \$_t \\ PI_t \\ PX_t \\ PM_t \\ B_t \end{bmatrix}$$

$$= \begin{bmatrix} (1+\theta_1B)\ s_t & pi_t & (1-\theta_1B)px_t & (1-\theta_1B^3)pm_t & (1-\theta_1B)\ b_t \end{bmatrix}'$$

Table 4 - $ARMAX_m$ Estimates

| PARAMETER | ESTIMATE | STANDARD ERROR | T-STATISTIC |
|---|---|---|---|
| 013 | -0.6524838 | 0.2351878 | -2.774310 |
| 0013 | 0.6553210 | 0.2443427 | 2.681974 |
| 00013 | 1.115592 | 0.2419704 | 4.610446 |
| 015 | -5.532663 | 2.517155 | -2.197983 |
| 00015 | 7.623234 | 2.637581 | 2.890236 |
| TH1 | 0.2312764 | 0.8226938E-01 | 2.811209 |
| 021 | 0.9352731E-02 | 0.2562426E-02 | 3.649952 |
| 00021 | 0.5521054E-02 | 0.2387180E-02 | 2.312793 |
| 024 | -0.6987430E-02 | 0.3600444E-02 | -1.940713 |
| 0024 | 0.9785057E-02 | 0.3682345E-02 | 2.657289 |
| PH2 | 0.5032622 | 0.6310806E-01 | 7.974610 |
| 031 | 0.8278838E-01 | 0.2230689E-01 | 3.711336 |
| 0031 | 0.3819698E-01 | 0.2058681E-01 | 1.855410 |
| 032 | 0.8135166 | 0.4851935 | 1.676685 |
| 0032 | 0.6496270 | 0.4864492 | 1.335447 |
| 034 | 0.1144679 | 0.3249124E-01 | 3.523037 |
| 0034 | -0.7307774E-01 | 0.3058622E-01 | -2.389237 |
| 00034 | -0.1005723 | 0.3153927E-01 | -3.188797 |
| TH3 | -0.3251419 | 0.8111832E-01 | -4.008243 |
| 041 | 0.8088264E-01 | 0.4839224E-01 | 1.671397 |
| 0041 | 0.7324786E-01 | 0.5049240E-01 | 1.450671 |
| 042 | 3.669557 | 1.315529 | 2.789416 |
| 0042 | -2.152577 | 1.105723 | -1.946759 |
| 0043 | 0.3761461 | 0.1563639 | 2.405582 |
| 045 | 6.491417 | 1.630439 | 3.981393 |
| 0045 | -4.783293 | 1.608308 | -2.974116 |
| TH4 | -0.2432166 | 0.8285714E-01 | -2.935373 |
| 053 | -0.1342656E-01 | 0.6070171E-02 | -2.211892 |
| 0053 | 0.2343000E-01 | 0.6060570E-02 | 3.865973 |
| 00053 | -0.1855719E-01 | 0.6059525E-02 | -3.062482 |
| 000053 | 0.2264843E-01 | 0.6218728E-02 | 3.641972 |
| 054 | -0.8565500E-02 | 0.2856343E-02 | -2.998765 |
| 0054 | 0.6706758E-02 | 0.2887536E-02 | 2.322658 |
| TH5 | -0.5505129 | 0.8373142E-01 | -6.574747 |

S

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 622.38114 | 6.64996 | 40.38554 | 92.76922 | 2.48058 |
| 2 | 6.64996 | 0.78331 | 2.32281 | 3.29266 | -0.03725 |
| 3 | 40.38554 | 2.32281 | 51.31809 | 30.62827 | 0.15928 |
| 4 | 92.76922 | 3.29266 | 30.62827 | 242.55711 | -1.07578 |
| 5 | 2.48058 | -0.03725 | 0.15928 | -1.07578 | 0.37412 |

*(4.5) Check of the Equivalence PCCV-CCV*

Writing the $ARMAX_m$ system in two-stage form we have

$$\dot{\Phi}(B)\, \mathbb{z}_t = \dot{\Theta}(B)\, \mathbb{u}_t$$

$$[\mathbb{I} - \Omega(B)]\, \mathbb{u}_t = \mathbb{e}_t$$

Now, to check empirically that the second equation admits the inversion $\mathbb{u}_t = [\mathbb{I}+\Omega(B)]\, \mathbb{e}_t$ (as a consequence of the equivalence PCCV-CCV), we must show that the estimation of the two models below provides equivalent results

$(AR_m^*)$ $\qquad \mathbb{u}_t = \Omega(B)\, \mathbb{u}_t + \mathbb{e}_t$

$(MA_m^*)$ $\qquad \mathbb{u}_t = \Omega(B)\, \mathbb{e}_t + \mathbb{e}_t$

The estimation of the first model yielded the results of Table 5. As for the second, after 6 iterations with stepsize 1, using as initial values the previous estimates, we have Table 6 which is very similar to Table 5 .

Table 5 - $AR_m^*$ Estimates

| PARAMETER | ESTIMATE | STANDARD ERROR | T-STATISTIC |
|---|---|---|---|
| 013 | -0.5685276 | 0.2359956 | -2.409061 |
| 0013 | 0.8119345 | 0.2437529 | 3.330974 |
| 00013 | 0.7504213 | 0.2485825 | 3.018801 |
| 015 | -4.804038 | 3.052745 | -1.573678 |
| 00015 | 7.499683 | 3.303831 | 2.269996 |
| 021 | 0.9894092E-02 | 0.2401165E-02 | 4.120538 |
| 00021 | 0.4007384E-02 | 0.2422164E-02 | 1.654464 |
| 024 | -0.8580347E-02 | 0.3569076E-02 | -2.404081 |
| 0024 | 0.9970648E-02 | 0.3765353E-02 | 2.647998 |
| 031 | 0.8068983E-01 | 0.1978320E-01 | 4.078705 |
| 0031 | 0.8124593E-01 | 0.1902123E-01 | 4.271329 |
| 032 | 1.388725 | 0.5920565 | 2.345595 |
| 0032 | 1.559550 | 0.6260811 | 2.490972 |
| 034 | 0.1216645 | 0.3163543E-01 | 3.845828 |
| 0034 | -0.2124893E-01 | 0.2972966E-01 | -0.7147386 |
| 00034 | -0.1179964 | 0.3165573E-01 | -3.727490 |
| 041 | 0.9141590E-01 | 0.4911557E-01 | 1.861241 |
| 0041 | 0.1343702 | 0.4845812E-01 | 2.772915 |
| 042 | 2.794655 | 1.568114 | 1.782177 |
| 0042 | -2.513924 | 1.358737 | -1.850192 |
| 0043 | 0.4657817 | 0.1564047 | 2.978055 |
| 045 | 6.085161 | 1.905692 | 3.193150 |
| 0045 | -5.329054 | 1.998529 | -2.666489 |
| 053 | -0.1491669E-01 | 0.6142348E-02 | -2.428500 |
| 0053 | 0.1620151E-01 | 0.6087920E-02 | 2.661256 |
| 00053 | -0.9143360E-02 | 0.6229507E-02 | -1.467750 |
| 000053 | 0.2049646E-01 | 0.6257042E-02 | 3.275742 |
| 054 | -0.8863305E-02 | 0.2845666E-02 | -3.114668 |
| 0054 | 0.4830612E-02 | 0.2901004E-02 | 1.665152 |

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| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 631.47999 | 6.41294 | 40.60615 | 112.27674 | 2.56567 |
| 2 | 6.41294 | 0.79427 | 2.67697 | 3.65158 | -0.06033 |
| 3 | 40.60615 | 2.67697 | 55.32032 | 36.62695 | 0.26424 |
| 4 | 112.27674 | 3.65158 | 36.62695 | 265.57731 | -0.90807 |
| 5 | 2.56567 | -0.06033 | 0.26424 | -0.90807 | 0.39353 |

Table 6 - $MA_m^*$ Estimates

| PARAMETER | ESTIMATE | STANDARD ERROR | T-STATISTIC |
|---|---|---|---|
| 013 | -0.6436207 | 0.3138530 | -2.050707 |
| 0013 | 1.090947 | 0.3164224 | 3.447756 |
| 00013 | 0.5432337 | 0.3168897 | 1.714267 |
| 015 | -5.907763 | 3.325947 | -1.776265 |
| 00015 | 7.647275 | 3.644263 | 2.098442 |
| 021 | 0.1058725E-01 | 0.2631262E-02 | 4.023640 |
| 00021 | 0.5853604E-02 | 0.2656674E-02 | 2.203358 |
| 024 | -0.9297268E-02 | 0.4018168E-02 | -2.313808 |
| 0024 | 0.1398902E-01 | 0.4148023E-02 | 3.372454 |
| 031 | 0.9798148E-01 | 0.2160122E-01 | 4.535924 |
| 0031 | 0.9147803E-01 | 0.2085136E-01 | 4.387150 |
| 032 | 0.7903036 | 0.6234725 | 1.267584 |
| 0032 | 1.270684 | 0.6468900 | 1.964297 |
| 034 | 0.1763749 | 0.3480377E-01 | 5.067696 |
| 0034 | -0.1490736E-01 | 0.3277734E-01 | -0.4548069 |
| 00034 | -0.1367913 | 0.3329693E-01 | -4.108225 |
| 041 | 0.1716409 | 0.5723436E-01 | 2.998914 |
| 0041 | 0.1418808 | 0.5242886E-01 | 2.706158 |
| 042 | 1.324075 | 1.696937 | 0.7802732 |
| 0042 | -2.878788 | 1.576186 | -1.826427 |
| 0043 | 0.6331799 | 0.2059767 | 3.074036 |
| 045 | 4.026279 | 2.075396 | 1.940005 |
| 0045 | -3.656133 | 2.252276 | -1.623306 |
| 053 | -0.1625040E-01 | 0.8296809E-02 | -1.958633 |
| 0053 | 0.1627930E-01 | 0.8455748E-02 | 1.925235 |
| 00053 | -0.1016059E-01 | 0.8339731E-02 | -1.218335 |
| 000053 | 0.1419680E-01 | 0.8427488E-02 | 1.684583 |
| 054 | -0.9463638E-02 | 0.3336303E-02 | -2.836564 |
| 0054 | 0.4851315E-02 | 0.3329529E-02 | 1.457057 |

S

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 677.49405 | 7.08519 | 60.19572 | 86.30623 | 0.77371 |
| 2 | 7.08519 | 0.75649 | 2.09254 | 2.53045 | -0.00933 |
| 3 | 60.19572 | 2.09254 | 47.71510 | 26.35743 | 0.57660 |
| 4 | 86.30623 | 2.53045 | 26.35743 | 267.37333 | -0.79684 |
| 5 | 0.77371 | -0.00933 | 0.57660 | -0.79684 | 0.40977 |

*(4.6) Parametric Comparisons*

Synthetic comparisons between the various models can be obtained through the classical likelihood ratio statistic on the generalized variances. Under the assumption that the $ARMA_m$ is a subclass of the more general TFS class (which has uncostrained spectral density matrix), we define the statistic

$$\hat{U}(n_1,_2) = \ln( |\hat{\Sigma}_{ARMA}|^{n_1} / |\hat{\Sigma}_{TFS}|^{n_2}) \xrightarrow{H_0} \chi^2(d)$$

where $(n_1,n_2)$ are the number of estimated residuals, and (d) the difference of the number of parameters in the two models .

Principal results of the parametric analysis are reported in Table 7. There, (N) is the number of parameters in the model , the (d) values in brackets are computed on the significant estimates, finally $RAR_m$ is TFS in simplified AR-form.

Table 7 - Parametric Comparisons

| Model | $\lvert \hat{\Sigma} \rvert$ | n | N | Pair | $\hat{U}$ | d | $\chi^2(1\%)$ |
|---|---|---|---|---|---|---|---|
| 1 $ARMA_m$ | 3 942 899 | 154 | 15 | (1,2) | 125.3 | 15 | 30.6 |
| 2 TFS | 1 731 615 | " | 30 | (2,3) | 4.8 | (2) | 9.2 |
| 3 $RAR_m$ | 1 786 775 | " | 30 | (2,4) | 17.5 | 4 | 13.3 |
| 4 $ARMAX_m$ | 1 545 809 | " | 34 | (4,5) | 6.2 | (1) | 6.6 |
| 5 $AR^*_m$ | 1 608 769 | " | 34 | (5,6) | 14.9 | (3) | 11.3 |
| 6 $MA^*_m$ | 1 772 543 | " | 34 | | | | |

On the basis of these results we briefly conclude that:

a) the rational TFS structure is effectively much more powerful than the linear $ARMA_m$ (see U(1,2)), and can be successfully identified in a disaggregate way;

b) in situations of adequate stationarity the polynomial orthogonal approximation holds (see U(2,3);

c) the $ARMAX_m$ reunification provides the best solution also in terms of speed of convergence and flexibility, it belongs however to the TFS class (U(2,4));

d) as a consequence of the polynomial orthogonality, simultaneous and sequential filtering are equivalent (see U(4,5)) ;

e) finally,for whitened series $AR_m$ and $MA_m$ representations tend to be exchangeable. That is to say that owing to the equivalence PCCV-CCV one may find a MA representation avoiding algebraic inversion of polynomial matrices .

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